Seat No. ______

[111]

No. of printed pages: 2

Sardar Patel University

M.Sc. (Mathematics) Semester - I Examination

Monday, 29th October, 2018

PS01CMTH24, Linear Algebra

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: 1. Figures to the right indicate marks of the respective question.
2. Assume standard notations and usual operations wherever applicable.

Q-1 Write the most appropriate **option number only** for each of the following question. [08]

1. Let V be a vector space over F such that $\dim \hat{\hat{V}} = 4$. Then $\dim A(V) =$ ____.
   (a) 2 (b) 4 (c) 8 (d) 16
2. ____ is a vector space over $\mathbb{C}$ with componentwise addition and scalar multiplication.
   (a) $\mathbb{R}\times\mathbb{R}$ (b) $\mathbb{C}\times\mathbb{R}$ (c) $\mathbb{C}\times\mathbb{C}$ (d) None of these
3. Let V be a finite-dimensional vector space over F and $T\in A(V)$. If 0 is a characteristic root of T, then T must be ____.
   (a) regular (b) singular (c) one-one (d) nilpotent
4. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as $T(x,y) = (iy, ix)$, $x, y \in \mathbb{C}$. Then the minimal polynomial for T is ____.
   (a) $1-x^2$ (b) $1+x^2$ (c) $1-x^4$ (d) $1+x^4$
5. Let V be a vector space over F and $T\in A(V)$. If W is a subspace of V invariant under T^2, then, W is invariant under ____.
   (a) T (b) T^3 (c) T^4 (d) none of these
6. The number of all possible Jordan forms of a 4×4 nilpotent matrix is ____.
   (a) 1 (b) 4 (c) 5 (d) 7
7. Let $A, B \in M_2(\mathbb{R})$ such that $\mathrm{tr}(A) = \mathrm{tr}(B) = 4$. Then $tr(AB) =$ ____.
   (a) 4 (c) 16
   (b) 8 (d) cannot be determined
8. Let $A\in M_6(\mathbb{C})$ be regular. Then $\det(iA)\det(A^{-1}) =$ ____.
   (a) -1 (b) 1 (c) i (d) $-i$

Q-2 Attempt *Any Seven* of the following: [14]

(a) Check if the set $\{(3, 5i, -7), (8i, -3, 2i), (18, -i, 20)\}$ is linearly independent over $\mathbb{C}$.

(b) Let U and W are subspaces of a vector space V. Show that $W^0 \subset U^0$ if $U \subset W$.

(c) Let $\mathscr{A}$ be an algebra over F with unit element e such that $\dim \mathscr{A} = m$. Show that every element of $\mathscr{A}$ satisfies a non-trivial polynomial over F of degree at most m.

(d) Let $F_n[x]$ denote the set of polynomials over F of degree at most n. Is $F_n[x]$ an algebra over F? Justify.

(e) Let V be a vector space over F and $S, T \in A(V)$ be nilpotent. Show that ST is nilpotent if $ST = TS$.

(f) Let V be a vector space over a field F and $T\in A(V)$. Show that $\ker(T)$ is an invariant subspace of V under T.

(g) For $A, B \in M_n(\mathbb{R})$, show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(h) Let F be a field and $A \in M_n(F)$ be invertible. Then show that $\det(A^{-1}) = \frac{1}{\det(A)}$.

(i) Compute the inertia of the quadratic form: $x^2 + y^2 + z^2$.

Q-3 (a) Let V be a finite-dimensional vector space over a field F and W be a subspace of V. Show that $\dim V/W = \dim V - \dim W$. [06]

(b) Let F be the field of all real numbers and let V be the set of all sequences $(a_1, a_2, \ldots)$, $a_i \in F$, where equality, addition and scalar multiplication are defined componentwise. Prove that V is a vector space over F. [06]

OR

(b) Let V be a finite-dimensional vector space over F. If A and B are subspaces of V, then show that $(A + B)/B$ is isomorphic to $A/(A \cap B)$. [06]

Q-4 (a) Let $\mathscr{A}$ be an algebra over a field F. Show that $\mathscr{A}$ is isomorphic to a subalgebra of $A(\mathscr{A})$. [06]

(b) Let V be vector space of polynomials of degree 3 or less over F. Let $T \in A(V)$ be defined by $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2$. Find the matrix A of T in the basis $v_1 = 1, v_2 = x, v_3 = x^2, v_4 = x^3$ and the matrix B of T in the basis $u_1 = 1, u_2 = 1 + x, u_3 = 1 + x^2, u_4 = 1 + x^3$. Hence, find a matrix C such that $C^{-1}AC = B$. [06]

OR

(b) Let V be a vector space over F and $T \in A(V)$. Show that characteristic vectors corresponding to distinct characteristic roots of T are linearly independent. [06]

Q-5 (a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Prove that the invariants of T are unique. [06]

(b) Find all possible Jordan forms for 6×6 matrices having $(x-2)^2(x-5)^2$ as minimal polynomial. [06]

OR

(b) Let V be a finite-dimensional vector space over F and $T \in A(V)$. Let W be a subspace of V invariant under T. Let $\bar{T} : V/W \to V/W$ be defined by $\bar{T}(v + W) = T(v) + W$. Show that $\bar{T}$ is well-defined and $\bar{T} \in A(V/W)$. Also show that minimal polynomial for $\bar{T}$ over F divides minimal polynomial for T over F. [06]

Q-6 (a) For $A, B \in M_n(F)$, show that $\det(AB) = \det(A)\det(B)$. [06]

(b) i. Prove or disprove: If $A \in M_n(\mathbb{R})$ such that $\operatorname{tr}(A) = 0$, then A is nilpotent. [02]

ii. Use Cramer's rule to solve the following system in the real field. [04]

$$\begin{aligned} x + y + z &= 1, \\ 2x + 3y + 4z &= 1, \\ x - y - z &= 0. \end{aligned}$$

OR

(b) Reduce the quadratic equation $3x^2 + 2xy + 3y^2 = 8$ into standard form by finding an orthogonal matrix P. Hence determine the conic section represented by it. [06]

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