[117] SEAT No.________ **No. of Printed Pages : 2**

Sardar Patel University

M.Sc. (Mathematics) External Examination 2018;
Code:- PS01CMTH23 : Subject :- Functions of Several Real Variables;
Date: 24-10-2018, Wednesday; Time- 10.00 am to 01.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x = (\sqrt{\pi}, 0, -1)$ and $y = (0, -\sqrt{e}, 1)$. Then $\|x + y\| =$

(a) $\sqrt{\pi + e}$ (b) $\pi + e$ (c) $\sqrt{\pi + e + 1}$ (d) $\pi + e + 1$

(ii) Which of the following map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is linear?

(a) $T(x) = x_1x_2$ (b) $T(x) = x_1 + x_2$ (c) $T(x) = x_1 + 2$ (d) $T(x) = 1 + x_2$

(iii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = \sin(x) + x$ $(x \in \mathbb{R})$. Then $Df(0)(x) =$

(a) $\cos(x)$ (b) x (c) $2x$ (d) $3x$

(iv) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_xf(a)$ exists for all $x \in \mathbb{R}^n$. Then

(a) f is continuously differentiable at a
(b) f is differentiable at a
(c) f is continuous at a
(d) $D_jf(a)$ exists for all $1 \le j \le n$

(v) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined as $f(x) = e^{x_1}$. Then $Df(0) =$

(a) 1 (b) π_1 (c) $e\pi_1$ (d) e^{π_1}

(vi) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at 0. Then which of the following is not true?
(a) f is continuous at 0
(b) Partial derivatives of f exist at 0
(c) f is continuously differentiable at 0
(d) Directional derivatives of f exists at 0

(vii) Let $S \in \mathcal{T}^1(V)$ and $T \in \mathcal{T}^5(V)$. Then $S \otimes T$ belongs to

(a) $\mathcal{T}^1(V)$ (b) $\mathcal{T}^5(V)$ (c) $\mathcal{T}^6(V)$ (d) none

(viii) Let π_1 and π_2 be the projection maps on $\mathbb{R}^2$. Then $\pi_1 \wedge \pi_2 =$

(a) $\pi_1 \otimes \pi_2 + \pi_2 \otimes \pi_1$ (b) $\pi_1 \otimes \pi_2 - \pi_2 \otimes \pi_1$ (c) $\pi_1 \otimes \pi_2$ (d) $\pi_2 \otimes \pi_1$

Q.2 Attempt any **seven.** [14]

(i) Prove that $\|x + y\| \le \|x\| + \|y\|$ $(x, y \in \mathbb{R}^n)$.
(ii) Prove that every norm preserving linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is inner product preserving.
(iii) Prove that every linear map is differentiable. What will be its derivation?
(iv) State the chain rule.
(v) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable, then prove that its component f^i is also differentiable.
(vi) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at a. Prove that $D_{sx}f(a) = sD_xf(a)$ $(s \in \mathbb{R}; x \in \mathbb{R}^n)$.
(vii) Let $T \in \mathcal{T}^2(\mathbb{R}^2)$ be defined as $T(x, y) = x_1y_1$. Find Alt(T).
(viii) Define vector field and k-form on $\mathbb{R}^n$.
(ix) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Define df and prove that it is 1-form on $\mathbb{R}^n$.

(P.T.O.)

Q.3

(a) Let $x, y \in \mathbb{R}^n$. Prove that $|\langle x, y\rangle| = \|x\|\|y\|$ iff x and y are dependent. [6]

(b) Let $A \subset \mathbb{R}^n$, let $f : A \longrightarrow \mathbb{R}$ be a bounded function, and let $a \in A$. Define $o(f; a)$. Then prove that f is continuous at a if and only if $o(f; a) = 0$. [6]

OR

(b) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear. Define $\|T\|$ and prove that $\|T(x)\| \leq \|T\|\|x\|$ $(x \in \mathbb{R}^n)$. [6]

Q.4

(a) If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then prove that there exists unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that [6]

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$

(b) Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Then prove that $f + g$ and fg are differentiable. [6]

OR

(b) Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ as $f(x) = (x_1 x_3, \sinh(x_2))$ $(x \in \mathbb{R}^3)$ and $a = (-1, 0, 2)$. Is f differentiable at a? If yes, then find its derivation $Df(a)$. [6]

Q.5

(a) Define the Jacobian matrix $f'(a)$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$. Then prove that $f'(a) = [f^{1'}(a), \ldots, f^{m'}(a)]$. [6]

(b) Prove that a continuously differentiable function is differentiable. [6]

OR

(b) Define $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ as $f(x) = (x_1^{x_2}, \cos(x_3 x_4), x_2 + x_3)$. Let $a = (2, 3, 1, \pi)$. Find both $f'(a)$ and $Df(a)$. [6]

Q.6

(a) Let V be a vector space with $\dim(V) = n$ and $k \in \mathbb{N}$. Prove that the $\dim(\mathcal{T}^k(V)) = n^k$. [6]

(b) Let $S \in \mathcal{T}^k(V)$ such that $\mathrm{Alt}(S) = 0$ and $T \in \mathcal{T}^\ell(V)$. Prove that $\mathrm{Alt}(S \otimes T) = 0$. [6]

OR

(b) Let $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. Prove that $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$. [6]

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