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Seat No. ________

[110]

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Sardar Patel University

M.Sc. (Mathematics) Semester - I Examination

Monday, 29th October, 2018

PS01CMTH04, Linear Algebra

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: 1. Figures to the right indicate marks of the respective question.
2. Assume standard notations and usual operations wherever applicable.

Q-1 Write the most appropriate **option number only** for each of the following question. [08]

1. Let V be a vector space over F such that $\dim \hat{\hat{V}} = 4$. Then $\dim V =$ ____.
   (a) 16 (b) 8 (c) 2 (d) 4
2. Let $V = \mathbb{C} \times \mathbb{R}$. Then $\dim V$ over $\mathbb{R}$ is ____.
   (a) 1 (b) 2 (c) 3 (d) 4
3. Let V be a finite-dimensional vector space over F and $S, T \in A(V)$. If T is regular, then $r(ST)$ ____.
   (a) $= r(T)$ (b) $= r(S)$ (c) $< r(TS)$ (d) $\geq r(T)$
4. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as $T(x,y) = (ix, iy)$, $x, y \in \mathbb{C}$. Then the minimal polynomial for T is ____.
   (a) $1 - x^2$ (b) $1 + x^2$ (c) $1 - x^4$ (d) $1 + x^4$
5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (0, 0, x_1)$, $x_1, x_2, x_3 \in \mathbb{R}$. Then the invariants of T are ____.
   (a) $1, 1, 1$ (b) 3 (c) $2, 1$ (d) none of these
6. Let V be a vector space and $T \in A(V)$ be nilpotent. Then $I - T$ is ____.
   (a) nilpotent (b) singular (c) regular (d) 0
7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (z, y, x)$, $x, y, z \in \mathbb{R}$. Then $\operatorname{tr}(T) =$ ____.
   (a) 3 (b) 2 (c) 0 (d) 1
8. Let $A \in M_3(\mathbb{R})$. Then $\det(2A^2) =$ ____.
   (a) $8\,(\det(A))^2$ (b) $2\,(\det(A))^2$ (c) $4\,(\det(A))^2$ (d) $64\,(\det(A))^2$

Q-2 Attempt *Any Seven* of the following: [14]

(a) Check if the set $\{(1,1,0,0), (0,1,-1,0), (0,0,0,3)\}$ is linearly independent over $\mathbb{R}$.

(b) Let W are subspaces of a vector space V over F. Define annihilator of W and show that it is a subspace of $\hat{V}$.

(c) Let V be a vector space over F and $T \in A(V)$. If $\lambda \in F$ is a characteristic root of T, then show that $T - \lambda I$ is singular

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x, x + y, x + y + z)$, $x, y, z \in \mathbb{R}$. Find the matrix of T in the basis $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 1, 1)$.

(e) Let V be a vector space over a field F and $T \in A(V)$. Show that $\ker(T)$ is an invariant subspace of V under T.

(f) Let V be a vector space over F and $T \in A(V)$. If T is nilpotent, then show that 0 is a characteristic root of T.

(P.T.O.)

(g) Let V be a vector space over F and $S, T \in A(V)$. If S is nilpotent and $ST = TS$, then show that ST is nilpotent.

(h) Show that there does not exist matrices A, B in $M_n(\mathbb{R})$ such that $AB - BA = I$.

(i) Write the symmetric matrix associated to the quadratic form $2xy + 2yz + 2zx$.

Q-3 (a) Let V be a finite-dimensional vector space over F. Show that V is isomorphic to $\hat{\hat{V}}$. [06]

(b) Let U and V be two vector spaces over a field F and $T : U \to V$ be an onto homomorphism. Show that $U/\ker T$ is isomorphic to V. [06]

OR

(b) Let F be any field. Show that F^n, the set of n-tuples in F, is a vector space over F with componentwise addition and scalar multiplication. [06]

Q-4 (a) Let V be an n-dimensional vector space over F. Show that there is an algebra isomorphism from $A(V)$ onto $M_n(F)$. [06]

(b) Let V be a 2-dimensional vector space over $\mathbb{R}$ with basis $\{v_1, v_2\}$. Let $T \in A(V)$ be defined by $Tv_1 = v_1 + v_2$ and $Tv_2 = v_1 - v_2$. Find the characteristics roots and corresponding characteristics vectors for T. [06]

OR

(b) Let V be a finite-dimensional vector space over F and $T \in A(V)$. If T is singular, then show that there exists $S \in A(V)$, $S \neq 0$ such that $TS = ST = 0$. [06]

Q-5 (a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Prove that the invariants of T are unique. [06]

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$. If all the characteristic roots of T are in F, then show that there is a basis of V with respect to which the matrix of T is upper triangular. [06]

OR

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be nilpotent linear transformation defined by $T(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, 0)$, for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Find the invariants of T and hence find a basis of $\mathbb{R}^4$ with respect to which the matrix of T has nilpotent canonical form. [06]

Q-6 (a) State and prove Cramer's rule. Hence show that for a field F if $A \in M_n(F)$ such that $\det(A) \neq 0$, then A is invertible. [06]

(b) i. Prove or disprove: If $A \in M_n(\mathbb{R})$ such that $\det(A) = 0$, then A is nilpotent. [02]

ii. Let F be a field of characteristic 0 and V be a finite-dimensional vector space over F. Let $S, T \in A(V)$ such that $S(ST - TS) = (ST - TS)S$. Then show that $ST - TS$ is nilpotent. [04]

OR

(b) Show that the determinant of a lower triangular matrix is the product of its entries on the main diagonal. [06]

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