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[116] SEAT No.________ No. of Printed Pages : 2

Sardar Patel University,

M.Sc. (Mathematics) External Examination 2018;
Code:- PS01CMTH03 : Subject :- Functions of Several Real Variables;
Date: 24-10-2018, Wednesday; Time- 10.00 am to 01.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x = (1, -1, 2)$ and $y = (-1, 1, -2)$. Then $\langle x, y \rangle =$

(a) 6 (b) -6 (c) 4 (d) -4

(ii) Let $x, y \in \mathbb{R}^n$ be orthogonal vectors. Then $\|x + y\|^2 =$

(a) $\|x\| + \|y\|$ (b) $(\|x\| + \|y\|)^2$ (c) $\|x\|^2 + \|y\|^2$ (d) $\|x\|\|y\|$

(iii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = x^3$ $(x \in \mathbb{R})$. Then $Df(2)(x) =$

(a) $12x$ (b) $18x$ (c) $24x$ (d) $30x$

(iv) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = 2x_1 e^{x_2}$. Then $D_1 f(0) =$

(a) -1 (b) 0 (c) 1 (d) 2

(v) Let $x = (1, 1)$. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(y) = y_1 + y_2$. Then $D_x f(0) =$

(a) 0 (b) 1 (c) 2 (d) 3

(vi) Let $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous at a. Then

(a) $D_j f(a)$ exists for all $1 \leq j \leq n$
(b) $D_x f(a)$ exists for all $x \in \mathbb{R}^n$
(c) $Df(a)$ exists
(d) none

(vii) Let $S \in \mathcal{T}^3(V)$ and $T \in \mathcal{T}^5(V)$. Then $S \otimes T$ belongs to

(a) $\mathcal{T}^8(V)$ (b) $\mathcal{T}^5(V)$ (c) $\mathcal{T}^3(V)$ (d) $\mathcal{T}^2(V)$

(viii) Let π_1 and π_2 be projection maps on $\mathbb{R}^2$. Then $\mathrm{Alt}(\pi_1 \otimes \pi_2) =$

(a) $\pi_1 \otimes \pi_2 - \pi_2 \otimes \pi_1$ (b) $\pi_1 \otimes \pi_2 + \pi_2 \otimes \pi_1$ (c) $\pi_1 \otimes \pi_2$ (d) none

Q.2 Attempt any **seven**. [14]

(i) Let $x, y \in \mathbb{R}^n$ be linearly dependent. Prove that $|\langle x, y \rangle| = \|x\|\|y\|$.
(ii) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous. Prove that fg is continuous.
(iii) Give an example of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which is continuous but not differentiable at a.
(iv) Define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as $f(x) = |x_1 x_2 x_3|$. Prove that f is differntiable at 0. What is $Df(0)$?
(v) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = e^{x_1}$. Find $Df(0)$.
(vi) Let $A \subset \mathbb{R}^n$ be open, $a \in A$, and $f : A \longrightarrow \mathbb{R}$. If f has maximum value at the point a and $D_i f(a)$ exists, then show that $D_i f(a) = 0$.
(vii) Is it true that if directional derivatives exist, then partial derivatives also exist? Justify.
(viii) Define vector field and k-form on $\mathbb{R}^n$.
(ix) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Define df and prove that it is 1-form on $\mathbb{R}^n$.

(P.T.O.)

Q.3

(a) Prove that $|\langle x,y\rangle| \leq \|x\|\|y\|$ $(x,y \in \mathbb{R}^n)$. [6]

(b) Let $A \subset \mathbb{R}^n$ be closed, $f : A \longrightarrow \mathbb{R}$ be bounded, and $\varepsilon > 0$. Then prove that the set $B = \{x \in A : o(f;x) \geq \varepsilon\}$ is closed in $\mathbb{R}^n$. [6]

OR

(b) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear. Prove that $\|T\| < \infty$ and $\|T(x)\| \leq \|T\|\|x\|$ $(x \in \mathbb{R}^n)$. [6]

Q.4

(a) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then prove that there exists a unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that [6]

$$\lim_{h\to 0} \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

(b) State and prove the chain rule. [6]

OR

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x) = \frac{|x_1 x_2|}{\|x\|}$ if $x \neq 0$ and $f(0) = 0$. Is the function f differentiable at 0? Justify your answer. [6]

Q.5

(a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as $f(x) = (\sin(x_1 \sin x_2), x_1^5 + 3x_2)$ and $a = (\pi, \pi/2)$. Find $f'(a)$ and $Df(a)$. [6]

(b) Prove that a continuously differentiable function is differentiable. [6]

OR

(b) Let $A = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } 0 < x_2 < x_1^2\}$ and $f = \chi_A$ be the characteristic. Prove that $D_x f(0)$ exists for all $x \in \mathbb{R}^2$, and f is not continuous at 0. [6]

Q.6

(a) Define $\text{Alt}(T)$. If $\omega \in \Lambda^k(V)$, then prove that $\text{Alt}(\omega) = \omega$. [6]

(b) Define wedge product. Prove that it is associative. Explicitly state results used in proof. [6]

OR

(b) Let $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. Prove that $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$. [6]

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