Seat No.: ________ Number of printed pages 2

[42]

# Sardar Patel University

## M. Sc. *(First Semester)* Examination

*Monday, 24$^{th}$ October 2016*

Course No. PS01CMTH04 : Linear Algebra

*Time: 10.00 a.m. to 01.00 p.m.* *Maximum marks: 70*

*Note:* (i) In this question paper, F denotes a field; R denotes the field of real numbers and $M_n(F)$ denotes the space of all square matrices of order n with entries in F.
(ii) Figures to the right indicate marks.

1. Fill up the gaps in the following: [8]
   i) Let V be a vector space over F and A, B be subspaces of V. Then $\dim(A+B) = \dim(A) + \dim(B)$, if ________.
   (a) $A = B$ (b) $A \cap B = \{0\}$ (c) $A + B = V$ (d) $A \cup B = V$.
   ii) Let $W = \{(x_1, x_2, x_3) \in R^3 : x_1 + 3x_2 + 4x_3 = 0\}$. Then the dim $(W^{\circ})$ = ____.
   (a) 0 (b) 2 (c) 1 (d) 3
   iii) Let V be any vector space over F and S, T $\in$ A(V) be such that ST = I. Then ________ is not true.
   (a) T is one to one (b) S is onto
   (c) S is regular (d) none of these
   iv) If $T : R^3 \to R^3$ is defined by $T(x_1, x_2, x_3) = (x_1, x_1, x_1)$, $(x_1, x_2, x_3) \in R^3$, then the rank of T is ________.
   (a) 3 (b) 2 (c) 0 (d) 1
   v) Define $T : R^4 \to R^4$ be nilpotent such that invariants of T are 1, 1, 1, 1. Then T is ________.
   (a) I (b) 0 (c) non zero (d) regular
   vi) Define $T : R^2 \to R^2$ by $T(x_1, x_2) = (-x_2, x_1)$, $(x_1, x_2) \in R^2$. Then p(x) = ________ is the minimal polynomial for T.
   (a) $1 - x^2$ (b) $x^2$ (c) $1 + x^2$ (d) none of these
   vii) Let *V* be a vector space of dimension 5 over F and T $\in$ A(V) such that rank(T) = 4. Then det(T) = ______.
   (a) 5 (b) non zero (c) 4 (d) 0
   viii) A, B $\in M_n(R)$. Then ________
   (a) $tr(\lambda A) = n\lambda\, tr(A)$ (b) $\det(A+B) = \det(A) + \det(B)$
   (c) $tr(A + B) = tr(A) + tr(B)$ (d) $\det(\lambda A) = \lambda \det(A)$

2. Answer any SEVEN of the following: [14]
   i) Let $V = R^2$. Find the dual basis of $v_1 = (2, 1)$, $v_2 = (1, 2) \in V$.
   ii) Let V be a finite dimensional vector space over F and S, T $\in$ A(V). Show that rank(ST) $\leq$ rank(T).
   iii) Suppose V is a vector space over F; T $\in$ A(V) satisfies $p(x) \in F[x]$. Let S $\in$ A(V) be regular. Show that $S^{-1}TS$ satisfies p(x).
   iv) Let $\alpha \in F$. Define $T : F^2 \to F^2$ by $T(x_1, x_2) = (\alpha x_2, x_1 + \alpha x_2)$, $(x_1, x_2) \in F^2$. Show that T is regular iff $\alpha \neq 0$.
   v) Let V be a vector space over F and S, T $\in$ A(V) be nilpotent such that ST = TS. Show that S + T is nilpotent.
   vi) Write all possible invariants of a nilpotent T $\in A(R^5)$.
   vii) Show that the set $\{A \in M_n(F) : \det(A) = 1\}$ is a group under matrix multiplication.
   viii) Let A, B $\in M_n(F)$. Show that tr(AB) = tr(BA).
   ix) Find the inertia of the quadratic equation $2x_1x_2 + 2x_1x_3 = 0$.

***P.T.O.***

3. a) Define internal direct sum and external direct sum of a vector space. Let V be a vector space over F and $V_1, V_2, \ldots, V_k$ be subspaces of V such that V is an internal direct sum of $V_1, V_2, \ldots, V_k$. Prove that V is isomorphic to the external direct sum of $V_1, V_2, \ldots, V_k$. [6]

b) i) Let V be a vector space over F; U and W be subspaces of V. Show that $(U+W)/W$ is isomorphic to $U/(U \cap W)$. [3]

ii) Let V be a vector space over R. Show that V cannot be represented as union of two proper subspaces of V. [3]

***OR***

b) Let V be a finite dimensional vector space over F and W be a subspace of V. Show that W is finite dimensional and $\dim V/W = \dim V - \dim W$. [6]

4. a) Let V be vector space over F and $T \in A(V)$. Show that the characteristic vectors corresponding to distinct characteristic roots of T are linearly independent. [6]

b) i) Let A be an n-dimensional algebra over F with unit element. Show that each element in A satisfies a polynomial in F[x] of degree $\leq n$. [3]

ii) Let V be a finite dimensional vector space over F, $T \in A(V)$ and $\lambda \in F$. Show that $\lambda$ is a characteristic root of T iff $\lambda$ is a root of the minimal polynomial for T. [3]

***OR***

b) Let V be a finite dimensional vector space over F and $T \in A(V)$. Show that T is regular iff the constant term of the minimal polynomial for T is nonzero. [6]

5. a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Show that the invariants of T are unique. [6]

b) Let V be a finite dimensional vector space over F and $T \in A(V)$. [6]

Let $p(x) = (q_1(x))^{l_1}(q_2(x))^{l_2} \ldots (q_k(x))^{l_k}$, (where $q_i(x)$, i=1,2,...,k are irreducible polynomials) be a minimal polynomial for T. Let $V_i = \ker (q_i(T))^{l_i}$ i = 1, 2,..., k. Show that each $V_i$ is a non-zero subspace of V which is invariant under T and $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$.

***OR***

b) Let V be a finite dimensional vector space over F; $T \in A(V)$ and $V_1$, $V_2$ be subspaces of V such that $V_1$, $V_2$ are invariant under T and $V = V_1 \oplus V_2$. Let $T_i = T/V_i$ and $p_i(x) \in F[x]$ be minimal polynomial for $T_i$, i=1, 2. Show that least common multiple of $p_1(x)$ and $p_2(x)$ is the minimal polynomial for T. [6]

6. a) Let $A, B \in M_n(F)$. Show that $\det(AB) = \det(A)\det(B)$. [6]

b) Let V be a finite dimensional vector space over F and $T \in A(V)$. Show that $tr(T^k) = 0$ for each k iff T is nilpotent. [6]

***OR***

b) i) State and prove Cramer's rule. [4]

ii) Let $A, B \in M_n(R)$. Show that $AB - BA \neq I$. [2]

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