(37) Seat NO: ________ **No. of Printed Pages : 2**

Sardar Patel University,

M.Sc. (Mathematics) External Examination 2016; I-SEM
Code:- PS01CMTH03 : Subject :- Functions of Several Real Variables;
Date: 19-10-2016, Wednesday; Time- 10.00 am to 1.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x = (1, -1, 2)$ and $y = (-1, 1, -2)$. Then $\langle x, y \rangle =$

(a) 6 (b) -6 (c) 4 (d) -4

(ii) Let $x, y \in \mathbb{R}^n$ be orthogonal vectors. Then $\|x + y\|^2 =$

(a) $\|x\| + \|y\|$ (b) $(\|x\| + \|y\|)^2$ (c) $\|x\|^2 + \|y\|^2$ (d) $\|x\|\|y\|$

(iii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(t) = 3t^3$. Then $Df(2) =$

(a) λ_9 (b) λ_{18} (c) λ_{27} (d) λ_{36}

(iv) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = 2x_1 e^{x_2}$. Then $D_2 f(0) =$

(a) -1 (b) 0 (c) 1 (d) 2

(v) Let $x = (1, 1)$. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(y) = y_1 + y_2$. Then $D_x f(0) =$

(a) 0 (b) 1 (c) 2 (d) 3

(vi) Let $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous at a. Then

(a) $D_j f(a)$ exists for all $1 \leq j \leq n$ (c) $Df(a)$ exists
(b) $D_x f(a)$ exists for all $x \in \mathbb{R}^n$ (d) none

(vii) Let $S \in \mathcal{T}^3(V)$ and $T \in \mathcal{T}^5(V)$. Then $S \otimes T$ belongs to

(a) $\mathcal{T}^8(V)$ (b) $\mathcal{T}^5(V)$ (c) $\mathcal{T}^3(V)$ (d) $\mathcal{T}^2(V)$

(viii) Let π_1 and π_2 be projection maps on $\mathbb{R}^2$. Then $\pi_1 \wedge \pi_2 =$

(a) $\pi_1 \otimes \pi_2 - \pi_2 \otimes \pi_1$ (b) $\pi_1 \otimes \pi_2 + \pi_2 \otimes \pi_1$ (c) $\pi_1 \otimes \pi_2$ (d) $\pi_2 \otimes \pi_1$

Q.2 Attempt any **seven**. [14]

(i) Prove that $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in \mathbb{R}^n)$.
(ii) Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuous at a. Prove that $f + g$ is also continuous at a.
(iii) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = x_1 + 3$. Prove that $Df(0) = \pi_1$ using the definition.
(iv) Let $a = (2, 1)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x) = x_1 x_2$. Find $Df(a)$.
(v) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a. Prove that each $f^i : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a.
(vi) Let $s \in \mathbb{R}$, $a, x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at a. Prove that $D_{sx} f(a) = s D_x f(a)$.
(vii) Give example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $D_x f(0)$ exist but $Df(a)$ does not exist.
(viii) Define an inner product on a vector space V.
(ix) Let $T \in \mathcal{T}^2(\mathbb{R}^2)$ be defined as $T(x, y) = x_1 y_2$. Find $\mathrm{Alt}(T)$.

(Continue on page-2)

(P.T.O.)

Q.3

(a) Let $x, y \in \mathbb{R}^n$. Prove that $|\langle x, y\rangle| = \|x\|\|y\|$ iff x and y are linearly dependent. [6]

(b) Let $A \subset \mathbb{R}^n$ be closed, let $f : A \longrightarrow \mathbb{R}$ be a bounded function, and let $\varepsilon > 0$. Then prove that the set $B = \{x \in A : o(f; x) \geq \varepsilon\}$ is closed in $\mathbb{R}^n$. [6]

OR

(b) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. Prove that there exists an $n \times m$ matrix A such that $T(x) = xA$ $(x \in \mathbb{R}^n)$. Further, if $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined as $T(x) = (x_1, 2x_2)$, then find A corresponding to T. [6]

Q.4

(a) State and prove the chain rule. [6]

(b) Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at a and $g(a) \neq 0$. Then prove that $\frac{f}{g}$ is also differentiable at a and $D(\frac{f}{g})(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}$. [6]

OR

(b) Let $a = (\pi, \frac{\pi}{2}, 1)$. Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as $f(x) = (x_1 \sin(x_2), \cos(x_2), x_3^2)$ $(x \in \mathbb{R}^3)$. Find the Jacobian matrix $f'(a)$ and the derivation $Df(a)$. [6]

Q.5

(a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$. Then prove that $D_x f(a) = Df(a)(x)$ and $D_{x+y} f(a) = D_x f(a) + D_y f(a)$. [6]

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$, let $1 \leq i \leq n$, and let $1 \leq j \leq m$. Then prove that the $(i, j)^{th}$-entry of the Jacobian matrix $f'(a)$ is exactly the i^{th}-entry of the Jacobian matrix $f^{i'}(a)$. [6]

OR

(b) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{|x_1 x_2|}{\|x\|} & (\text{if } x \neq 0) \\ 0 & (\text{if } x = 0) \end{cases}$$

Prove that $D_1 f(0) = D_2 f(0) = 0$ but $D_x f(0)$ does not exist for $x = (1, 1)$. [6]

Q.6

(a) Let $k \in \mathbb{N}$ and V be a vector space with dimension n. Prove that $\dim(\mathcal{T}^k(V)) = n^k$. [6]

(b) Let $S \in \mathcal{T}^k(V)$ such that $\mathrm{Alt}(S) = 0$ and $T \in \mathcal{T}^\ell(V)$. Prove that $\mathrm{Alt}(S \otimes T) = 0$. [6]

OR

(b) Define "k-form" on $\mathbb{R}^n$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable. Then prove that [6]

$$\widetilde{f}_{1*}(d\pi_i) = \sum_{j=1}^{n} D_j f^i \cdot d\pi_j \quad (1 \leq i \leq m).$$

THE END

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