Seat No. ______

[108]

No. of printed pages: 2

Sardar Patel University
M.Sc. (Mathematics) Semester - I Examination
Wednesday, 27th March, 2019
PS01CMTH24, Linear Algebra

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: 1. Figures to the right indicate marks of the respective question.
2. Assume usual/standard notations wherever applicable.

Q-1 Write the most appropriate **option only** for each of the following question. [08]

1. Let V be a vector space over F such that $\dim(\text{Hom}(V, F)) = 9$. Then $\dim V =$ ____.
   (a) 3 (b) 9 (c) 27 (d) 81
2. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x - y - z = 0\}$. Then $\dim W =$ ____.
   (a) 0 (b) 1 (c) 2 (d) 3
3. Let V be a finite-dimensional vector space and $S, T \in A(V)$ be such that $ST = I$. Then ____ is not true.
   (a) T is one-one (b) S is regular (c) S is onto (d) none of these
4. Let V be a vector space over F with $\dim(A(V)) = 4$. Then $\dim V =$ ____.
   (a) 16 (b) 8 (c) 4 (d) 2
5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (0, 0, z)$. Then invariants of T are ____.
   (a) 1, 1, 1 (b) 3 (c) 2, 1 (d) none of these
6. Let V be a vector space, $S \in A(V)$ be such that S is nilpotent. Then $S - I$ is ____.
   (a) nilpotent (b) regular (c) singular (d) zero map
7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (0, y, 0)$, $x, y, z \in \mathbb{R}$. Then $\text{tr}(T) =$ ____.
   (a) 0 (b) 1 (c) 2 (d) 3
8. Let $A \in M_n(\mathbb{R})$ with $\det(A) = 0$. Then A is a ____ matrix.
   (a) diagonal (b) regular (c) singular (d) nilpotent

Q-2 Attempt *any seven* of the following. [14]

(a) Verify if the set $\{(1, 2, 3), (1, 1, 0), (0, 1, 3)\}$ is linearly independent over $\mathbb{R}$.

(b) Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$. Is W a subspace of $\mathbb{R}^3$ over $\mathbb{R}$? Justify.

(c) Let V be a vector space over F and $T \in A(V)$. If $\lambda \in F$ is a characteristic root of T, then show that $T - \lambda I$ is singular

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x - y, y - z, z - x)$. Find matrix of T.

(e) Give an example of a nilpotent linear transformation with index of nilpotence 4.

(f) Write the Jordan matrix of a linear transformation having both characteristic polynomial and minimal polynomial as x^3.

(g) Let F be a field and $A, B \in M_n(F)$. Show that $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

(h) Show by giving an example that for $A, B \in M_3(\mathbb{R})$, $\det(A + B) \neq \det(A) + \det(B)$.

(i) Write the symmetric matrix associated to the following quadratic form:

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2x_1x_2 - 10x_1x_4 + 4x_3x_4.$$

Q-3 (a) Let U and V be two vector spaces over a field F and $T : U \to V$ be an onto homomorphism. Show that $U/\ker T$ is isomorphic to V. [06]

(b) Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $\{u_1, u_2, \ldots, u_m\}$ be a linearly independent set of V. Then show that $m \leq n$. [06]

OR

(b) Let V be a finite-dimensional vector space over F and W be a subspace of V. Show that $\dim W^0 = \dim V - \dim W$, where W^0 is the annihilator of W. [06]

Q-4 (a) Let V be an n-dimensional vector space over a field F. Show that $A(V)$ and $M_n(F)$ are isomorphic as algebras. [06]

(b) Let V be a vector space over F and $T \in A(V)$. If $\lambda_1, \lambda_2, \ldots, \lambda_k \in F$ are distinct characteristic roots of T and $v_1, v_2, \ldots, v_k$ are characteristic vectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively, then show that $v_1, v_2, \ldots, v_k$ are linearly independent. [06]

OR

(b) Let V be a vector space over F and $T \in A(V)$. Show that T is regular if and only if the constant term of the minimal polynomial for T is non-zero. [06]

Q-5 (a) Let V be a finite dimensional vector space over F and $T \in A(V)$. If all the characteristic roots of T are in F, then show that there is a basis of V with respect to which the matrix of T is upper triangular. [06]

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Prove that the invariants of T are unique. [06]

OR

(b) Let V be a finite dimensional vector space over F, $T \in A(V)$, and V_1 and V_2 be subspaces of V invariant under T such that $V = V_1 \oplus V_2$. Let $T_1 = T|_{V_1}$ and $T_2 = T|_{V_2}$. If minimal polynomials for T_1 and T_2 over F are $p_1(x)$ and $p_2(x)$ respectively, then show that the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$. [06]

Q-6 (a) For $A, B \in M_n(F)$, show that $\det(AB) = \det(A)\det(B)$. [06]

(b) Let F be a field of characteristic 0, V be a vector space over F and $T \in A(V)$. If $\operatorname{tr}(T^i) = 0$ for all $i \geq 1$ then show that T is nilpotent. [06]

OR

(c) i. Prove or disprove: For $A, B \in M_n(\mathbb{R})$, $\operatorname{tr}(A) = \operatorname{tr}(A^{-1})$. [02]

ii. State and prove Cramer's rule. [04]

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