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SEAT No.________

[92]

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Sardar Patel University, Department of Mathematics

M.Sc. (Mathematics) External Examination 2019;
Code:- PS01CMTH23 : Subject :- Functions of Several Real Variables;
Date: 25-03-2019, Monday; Time- 10.00 am to 01.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x = (\sqrt{2}, \pi, e)$ and $y = (0, -e, \pi)$. Then $\langle x, y \rangle =$

(a) 0 (b) 1 (c) 2 (d) 3

(ii) Let $x, y \in \mathbb{R}^n$ be orthogonal vectors. Then $\|x + y\| =$

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) $\sqrt{4}$

(iii) $\lim_{x \to 0} \left(\frac{\sin x}{x} + \frac{x}{\cos x}\right) =$

(a) 0 (b) 1 (c) ∞ (d) none

(iv) Let $a \in \mathbb{R}^n$ be fixed. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous at a. Then

(a) $Df(a)$ exists (b) $D_x f(a)$ exist (c) $D_j f(a)$ exist (d) none

(v) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined as $f(x) = |x_1| + |1 - x_2| + |1 + x_3|$. Then f is differentiable at

(a) $(0, 0, 0)$ (b) $(0, 1, 0)$ (c) $(0, 0, -1)$ (d) $(1, 2, 3)$

(vi) Let $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_x f(a)$ exists for all $x \in \mathbb{R}^n$. Then

(a) $Df(a)$ exists
(b) f is continuous at a
(c) $D_j f(a)$ exists for all $1 \leq j \leq n$
(d) f is continuously differentiable at a

(vii) Let $S \in \mathcal{T}^3(V)$ and $T \in \mathcal{T}^5(V)$. Then $S \otimes T$ belongs to

(a) $\mathcal{T}^{15}(V)$ (b) $\mathcal{T}^8(V)$ (c) $\mathcal{T}^5(V)$ (d) $\mathcal{T}^3(V)$

(viii) Let π_1 and π_2 be projection maps on $\mathbb{R}^5$. Then, for $x, y \in \mathbb{R}^5$, we have $(\pi_1 \otimes \pi_2)(x, y) =$

(a) $x_1 y_2$ (b) $x_2 y_1$ (c) $x_1 y_1$ (d) $x_2 y_2$

Q.2 Attempt any seven. [14]

(i) Prove that $\| x + y \| \leq \| x \| + \| y \|$ $(x, y \in \mathbb{R}^n)$.
(ii) Define the oscillation $o(f; a)$ of f at a.
(iii) Prove that every norm preserving linear map is inner product preserving.
(iv) Prove that every linear map is differentiable and its derivation is itself.
(v) Is it true that if all partial derivatives of $f : \mathbb{R}^n \to \mathbb{R}$ exist at a, then $Df(a)$ exists? Justify.
(vi) Let $a = (2, 2)$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x) = x_1 x_2$. Find $Df(a)$.
(vii) Let $s \in \mathbb{R}, a, x \in \mathbb{R}^n, f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at a. Prove that $D_{sx} f(a) = s D_x f(a)$.
(viii) Define tensor product and wedge product.
(ix) Define field and k-form.

(Continue on Page-2)

(P.T.O.)

Q.3

(a) Define the inner product on $\mathbb{R}^n$. Then state and prove the polarization identity. [6]

(b) Let $A \subset \mathbb{R}^n$, let $f : A \longrightarrow \mathbb{R}$ be a bounded function, and let $a \in A$. Then prove that f is continuous at a iff $o(f; a) = 0$. [6]

OR

(b) Let $A \subset \mathbb{R}^n$ be closed, $f : A \longrightarrow \mathbb{R}$ be bounded function, and $\varepsilon > 0$. Prove that the $\{x \in A : o(f; x) \geq \varepsilon\}$ is a closed set in $\mathbb{R}^n$. [6]

Q.4

(a) Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. Then prove that $f + g$ and fg are differentiable. [6]

(b) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = |x_1|\sqrt{|x_2|}$ $(x \in \mathbb{R}^2)$. Does $Df(0)$ exist? If yes, then find it. [6]

OR

(b) State and prove the chain rule. [6]

Q.5

(a) Find the derivation of $f(x) = (x_1 x_2, x_1 + x_2^2)$ at $(2, 3)$. [6]

(b) Prove that every continuously differentiable function is differentiable. [6]

OR

(b) Let $a = (\pi, \pi)$ and $f(x) = (x_1 \cos x_2, x_1 - x_2)$ $(x \in \mathbb{R}^2)$. First find the Jacobian matrix $f'(a)$ and then find $Df(a)$. [6]

Q.6

(a) Let $S \in \mathcal{T}^k(V)$ such that $\mathrm{Alt}(S) = 0$ and $T \in \mathcal{T}^\ell(V)$. Prove that $\mathrm{Alt}(S \otimes T) = 0$. [6]

(b) Let $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. Then prove that $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$. [6]

OR

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable. Then prove that [6]

$$\widetilde{f}_{1*}(d\pi_i) = \sum_{j=1}^{n} D_j f^i \cdot d\pi_j \quad (1 \leq i \leq m).$$

THE END

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