Seat No. _______

[107]

No. of printed pages: 2

Sardar Patel University

M.Sc. (Mathematics) Semester - I Examination

Wednesday, 27^{th} March, 2019

PS01CMTH04, Linear Algebra

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: 1. Figures to the right indicate marks of the respective question.
2. Assume usual/standard notations wherever applicable.

Q-1 Write the most appropriate **option only** for each of the following question. [08]

1. Let V be a vector space over F such that $\dim V = 9$. Then $\dim \widehat{V} =$ ____.
   (a) 81 (b) 27 (c) 9 (d) 3
2. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$. Then $\dim W^0 =$____.
   (a) 0 (b) 1 (c) 2 (d) 3
3. Let V be any vector space and $S, T \in A(V)$ with $ST = I$. Then ____ is not true.
   (a) T is one-one (b) S is regular (c) S is onto (d) none of these
4. Let V be a vector space over F with $\dim V = 4$. Then $\dim A(V) =$ ____.
   (a) 16 (b) 8 (c) 4 (d) 2
5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (z, 0, 0)$. Then invariants of T are ____.
   (a) 1, 1, 1 (b) 3 (c) 2, 1 (d) none of these
6. Let V be a vector space, $T \in A(V)$ be such that $I - T$ is nilpotent. Then T is ____.
   (a) nilpotent (b) regular (c) singular (d) identity
7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (0, y, 0)$, $x, y, z \in \mathbb{R}$. Then $\det(T) =$ ____.
   (a) 0 (b) 1 (c) 2 (d) 3
8. Let $A \in M_n(\mathbb{R})$ with non-zero determinant. Then A is a ____ matrix.
   (a) diagonal (b) regular (c) singular (d) nilpotent

Q-2 Attempt *any seven* of the following. [14]

(a) Verify if the set $\{(1, 1, 1), (1, 1, 0), (0, 0, 1)\}$ is linearly independent over $\mathbb{C}$.

(b) Let $W = \{(x, y, z) \in \mathbb{C}^3 \mid \operatorname{Re}(x + y + z) = 0\}$. Is W a subspace of $\mathbb{C}^3$ over $\mathbb{C}$? Why?

(c) Let V be a vector space over F and $T \in A(V)$. If $\lambda \in F$ is a characteristic root of T, then show that $T - \lambda I$ is singular

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (z, y, x)$. Find the matrix of T.

(e) Give an example of a nilpotent linear transformation with index of nilpotence 3.

(f) Let V be a vector space over F and $T \in A(V)$. If T is nilpotent, then show that 0 is a characteristic root of T.

(g) Let F be a field, $A, B \in M_n(F)$, and $\lambda \in F$. Show that $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr}(A)$.

(h) Show that there does not exist matrices A, B in $M_n(\mathbb{R})$ such that $AB - BA = I$.

(i) Write the symmetric matrix associated to the following quadratic form:

$$9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + 2x_2x_3.$$

(P.T.O.)

Q-3 (a) Let V be a finite-dimensional vector space over a field F and W be a subspace of V. Show that $\dim V/W = \dim V - \dim W$. [06]

(b) Define internal direct sum of vector spaces. Show that internal direct sum is isomorphic to external direct sum. [06]

OR

(b) Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. If $\{u_1, \ldots, u_m\}$ in V is linearly independent then show that $m \leq n$. [06]

Q-4 (a) Prove that an algebra $\mathcal{A}$ with unit element over a field F is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. [06]

(b) Let V be a vector space over F and $T \in A(V)$. Show that T is regular if and only if the constant term of the minimal polynomial for T is non-zero. [06]

OR

(b) Let V be a vector space over F and $T \in A(V)$. Let $B_1 = \{v_1, v_2, \ldots, v_n\}$ and $B_2 = \{w_1, w_2, \ldots, w_n\}$ be bases of V over F. If $m_1(T)$ and $m_2(T)$ are matrices of T with respect to B_1 and B_2 respectively, then show that $m_1(T)$ and $m_2(T)$ are similar. [06]

Q-5 (a) Let V be an n-dimensional vector space over a field F and $T \in A(V)$ be such that all its characteristic roots are in F. Prove that T satisfies a polynomial of degree n over F. [06]

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$. Let $p(x) = (q_1(x))^{l_1}(q_2(x))^{l_2} \cdots (q_k(x))^{l_k}$ be the minimal polynomial for T, where $q_i(x) \in F[x]$ is irreducible, $i = 1, \ldots, k$. Let $V_i = \ker(q_i(T))^{l_i}$. Show that each $V_i \neq \{0\}$. Assuming that V_i is invariant under T, show that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$. [06]

OR

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Prove that the invariants of T are unique. [06]

Q-6 (a) For $A, B \in M_n(F)$, show that $\det(AB) = \det(A)\det(B)$. [06]

(b) i. Show by giving an example that $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$ is not true. [02]

ii. State and prove Jacobson lemma. [04]

OR

(b) Let F be a field of characteristic 0, V be a vector space over F and $T \in A(V)$. If $\mathrm{tr}(T^i) = 0$ for all $i \geq 1$ then show that T is nilpotent. [06]

——— × ———