SEAT No.________

[917]

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Sardar Patel University, Department of Mathematics

M.Sc. (Mathematics) External Examination 2019;

Code:- PS01CMTH03 : Subject :- Functions of Several Real Variables;

Date: 25-03-2019, Monday; Time- 10.00 am to 01.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x = (\sqrt{2}, 1, e)$ and $y = (\sqrt{2}, 2, e)$. Then $\|x - y\| =$

(a) 1 (b) 2 (c) 3 (d) none

(ii) Let $x, y \in \mathbb{R}^n$. Then $|\langle x, y\rangle| \leq$

(a) $\sqrt{\|x\|\|y\|}$ (b) $\|x\|\|y\|$ (c) $\|x\|^2\|y\|^2$ (d) $\|x\|^3\|y\|^3$

(iii) Let $x, y \in \mathbb{R}^n$ be orthonormal vectors. Then $\|x + y\|^2 =$

(a) 1 (b) 2 (c) 3 (d) 4

(iv) Let $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_x f(a)$ exists for all $x \in \mathbb{R}^n$. Then

(a) f is continuous at a
(b) f is differentiable at a
(c) $D_j f(a)$ exists for all $1 \leq j \leq n$
(d) none

(v) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_j f(a)$ exists for all $1 \leq j \leq n$. Then f is ________ at a.

(a) Direc Diff. (b) Cont. Diff. (c) differentiable (d) None

(vi) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x) = \sqrt{|x_1 x_2|}$. Then

(a) f is continuous
(b) f is continuous only at origin
(c) f is differentiable
(d) f is differentiable only at origin

(vii) Let S and T be k-tensors on V. Then

(a) $S \otimes T = T \otimes S$
(b) $S + T = T + S$
(c) $S - T = T - S$
(d) None of these

(viii) The dimension of $\mathcal{T}^2(\mathbb{R}^4)$ is

(a) 2 (b) 4 (c) 8 (d) 16

Q.2 Attempt any seven. [14]

(i) Let $x, y \in \mathbb{R}^n$ be linearly dependent. Prove that $|\langle x, y\rangle| = \|x\|\|y\|$.
(ii) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Prove that $f + g$ is differentiable.
(iii) Give an example of a function which is continuous but not differentiable at a point a.
(iv) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = |x_1 x_2|$. Prove that $Df(0)$ exists.
(v) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = x_2 e^{x_1}$. Find $Df(0)$.
(vi) Let $A \subset \mathbb{R}^n$ be open, $a \in A$, and $f : A \longrightarrow \mathbb{R}$. If f has maximum value at the point a and $D_i f(a)$ exists, then show that $D_i f(a) = 0$.
(vii) Is it true that if partial derivatives exist, then directional derivatives also exist? Justify.
(viii) Is it true that $S \otimes T = T \otimes S$? Justify your answer.
(ix) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. If $df(p)(v_p) = Df(p)(v)$ $(p, v \in \mathbb{R}^n)$, then prove that $df(p)$ is a linear functional on $\mathbb{R}^n_p$ for each $p \in \mathbb{R}^n$.

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(P.T.O.)

Q.3

(a) Let $x, y \in \mathbb{R}^n$. Then prove that $| < x, y > | = \|x\|\|y\|$ iff x and y are dependent. [6]

(b) Let $A \subset \mathbb{R}^n$ be closed, $f : A \longrightarrow \mathbb{R}$ be a bounded function, and $\varepsilon > 0$. Then prove that the set $B = \{x \in A : o(f; x) \geq \varepsilon\}$ is closed in $\mathbb{R}^n$. [6]

OR

(b) Define norm preserving and inner product preserving linear map. Prove that both the definitions are equivalent. [6]

Q.4

(a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $a \in \mathbb{R}^n$. Define the derivation of f at a. If f is differentiable at a, then prove that there exists unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that [6]

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - \lambda(h)\|}{\|h\|} = 0.$$

(b) State and prove the chain rule. [6]

OR

(b) Find the derivation of $f(x) = (x_1 x_2, x_1 + x_2^2)$ at $(2, 3)$. [6]

Q.5

(a) Let $a = (\pi, \pi)$ and $f(x) = (x_1 \cos x_2, x_1 - x_2)$ $(x \in \mathbb{R}^2)$. First find the Jacobian matrix $f'(a)$ and then find $Df(a)$. [6]

(b) Prove that continuously differentiable function is differentiable. [6]

OR

(b) Give an example of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $D_j f(0)$ exists for all $j = 1, 2$ but f is not continuous at 0. [6]

Q.6

(a) Define Alt(T). If Alt$(S) = 0$, then prove that Alt$(S \otimes T) = 0$ for any tensor T. [6]

(b) Define the wedge product. Prove that it is associative but not commutative. [6]

OR

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Let $\widetilde{f}_{k*} : \Delta_{kF}(\mathbb{R}^m) \to \Delta_{kF}(\mathbb{R}^n)$ be defined as $\widetilde{f}_{k*}(\omega)(p) := \widetilde{f}^*_{pk}(\omega(f(p)))$ $(p \in \mathbb{R}^n; \omega \in \Delta_{kF}(\mathbb{R}^m))$. Then prove that $\widetilde{f}_{k*}$ is well-defined and it is a linear map. [6]

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