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Seat No. ______ No. of printed pages: 2

[34]

# SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - I Examination (NC)
Thursday, $19^{th}$ April, 2018
PS01CMTH24, Linear Algebra

**Time: 10:00 a.m. to 01:00 p.m.** **Maximum marks: 70**

Note: 1. Figures to the right indicate full marks of the respective question.
2. Assume standard notations wherever applicable.

**Q-1** Write the question number and appropriate **option number only** for each question. [8]

1. Let $V$ be any vector space over a field $F$. Let $W$ be a subspace of $V$ and $W^0$ be annihilator of $W$. If $\dim V = 5$, $\dim W^0 = 3$, then $\dim W =$ ______.
   (a) 1 (b) 2 (c) 3 (d) 5
2. For subspaces $U$ and $W$ of a vector space $V$ over $F$, ______ need not be a subspace of $V$.
   (a) $L(U) \cup L(W)$ (b) $U \cup W$ (c) $U \cap W$ (d) $L(U) \cap L(W)$
3. Let $V$ be a finite-dimensional vector space and $S \in A(V)$ be right invertible. Then ______.
   (a) $S$ is singular (b) $S$ is one-one (c) $S$ is 0 (d) $S$ is nilpotent
4. Let $V$ be a vector space over a field $F$ with $\dim V = n$. Then the algebra $\text{Hom}(V, V)$ over $F$ is isomorphic to ______
   (a) $F_n[x]$ (b) $M_n(F)$ (c) $F^n$ (d) $V$
5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then the index of nilpotence of $T$ is ______.
   (a) 0 (b) 1 (c) 2 (d) 3
6. Let $V$ vector space over $F$ and $T \in A(V)$ be nilpotent. Then $I + T^2 + 3T^3$ is ______.
   (a) nilpotent (b) singular (c) regular (d) none of these
7. Let $A \in M_n(\mathbb{C})$ with $\det(A) = -1$. Then $\det(A^{-1}) =$ ______.
   (a) 1 (b) $-1$ (c) $i$ (d) $-i$
8. Let $F$ be a field and $A \in M_n(F)$ be nilpotent. Then $\det(A^2) =$ ______.
   (a) 0 (b) 2 (c) 4 (d) none of these

**Q-2** Attempt ***Any Seven*** of the following: [14]

(a) Let $S$ be a non-empty subset of a vector space $V$ over $F$. Show that $L(S)$ is a subspace of $V$.

(b) Let $W$ be a subspace of a vector space over $F$. Define annihilator of $W$.

(c) Let $V$ be a finite dimensional vector space over $F$ and $T \in A(V)$. Show that a characteristic root of $T$ is a root of the minimal polynomial for $T$.

(d) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x_1, x_2, x_3) = (2x_1 + x_2, 2x_2 + x_3, 2x_3 + x_1)$, $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of $T$ with respect to standard basis of $\mathbb{R}^3$.

(e) Define nilpotent linear transformation on a vector space and give an example.

(f) Let $V$ be a vector space over a field $F$ and $T \in A(V)$ be nilpotent. Show that $I - T$ is invertible under $T$.

(g) Show that trace of a matrix is a linear operator on $M_n(F)$, where $F$ is a field.

(h) Show that similar matrices have the same determinant.

(i) Find the symmetric matrix associated with the quadratic form: $xy + yz + zx$.



(P.T.O.)

**Q-3** (a) Let $V$ be a vector space and $\{v_1, v_2, \ldots, v_n\}$ be a basis of $V$. If $\{w_1, w_2, \ldots, w_m\}$ is a linearly independent set of $V$, then show that $m \leq n$. [6]

(b) Let $V = U_1 \oplus \cdots \oplus U_n$ be the internal direct sum of $U_1, \ldots, U_n$. Show that $V$ is isomorphic to the external direct sum of $U_1, \ldots, U_n$. [6]

**OR**

(b) Let $V$ be a finite-dimensional vector space over a field $F$. If $W$ is a subspace of $V$ then show that $W$ is also finite-dimensional and $\dim V/W = \dim V - \dim W$. [6]

**Q-4** (a) Let $V$ be a vector space over a field $F$ and let $T \in A(V)$. Show that $T$ is invertible if and only if the constant term of the minimal polynomial for $T$ is non-zero. [6]

(b) Let $V$ be a finite-dimensional vector space over $F$ and $S, T \in A(V)$. Show that $r(ST) \leq r(T)$. Further if $S$ is invertible, then show that $r(ST) = r(TS) = r(T)$. [6]

**OR**

(b) Let $V$ be a finite-dimensional vector space over $F$. Show that $A(V)$ is closed under addition, multiplication, and scalar multiplication. [6]

**Q-5** (a) Let $V$ be a finite dimensional vector space over $F$ and $T \in A(V)$ be nilpotent. Then show that the invariants of $T$ are unique. [6]

(b) Let $V$ be a finite dimensional vector space over $F$, $T \in A(V)$, and $V_1$ and $V_2$ be subspaces of $V$ invariant under $T$ such that $V = V_1 \oplus V_2$. Let $T_1 = T\restriction_{V_1}$ and $T_2 = T\restriction_{V_2}$. If minimal polynomials for $T_1$ and $T_2$ over $F$ are $p_1(x)$ and $p_2(x)$ respectively, then show that the minimal polynomial of $T$ over $F$ is the least common multiple of $p_1(x)$ and $p_2(x)$. [6]

**OR**

(b) Let $V$ be a finite-dimensional vector space over $F$ and $T \in A(V)$. If $T$ has all its characteristic roots in $F$, then show that there is a basis of $V$ with respect to which the matrix of $T$ is upper triangular. [6]

**Q-6** (a) Prove that determinant of the product of two $n \times n$ matrices over a field $F$ is the product of their determinants. [6]

(b) Prove that interchanging two rows of a matrix changes the sign of its determinant. [6]

**OR**

(b) i. State and prove Cramer's rule. [4]

ii. For $A, B \in M_2(\mathbb{R})$, by giving an example, show that $\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$. [2]

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