SEAT No.________ No. of Printed Pages : 2

[39]

Sardar Patel University

M.Sc. (Mathematics) External Examination 2018;

Code:- PS01CMTH23 : Subject :- Functions of Several Real Variables;

Date: 12-04-2018, Thursday; Time- 10.00 am to 01.00 pm ; Max. Marks 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x, y \in \mathbb{R}^n$. Then

(a) $\langle x, y\rangle \geq 0$ (b) $\langle x, y\rangle = 0$ (c) $\langle x, y\rangle \leq 0$ (d) none

(ii) Which of the following is true?

(a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 0$ (b) $\lim\limits_{x\to 0} \frac{\cos x}{x} = 1$ (c) $\lim\limits_{x\to 0} x\cos(\frac{1}{x}) = 0$ (d) none

(iii) Let $a = (2,1)$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x) = x_1x_2$. Then $Df(a) =$

(a) $\pi_1 + \pi_2$ (b) $2\pi_1 + \pi_2$ (c) $\pi_1 + 2\pi_2$ (d) none

(iv) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$. Then which is false?

(a) $D_x f(a)$ exists for all $x \in \mathbb{R}^n$
(b) f is continuous at a
(c) $D_j f(a)$ exists for all $1 \leq j \leq n$
(d) All are false.

(v) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_x f(a)$ exists for all $x \in \mathbb{R}^n$. Then

(a) f is continuous at a
(b) $D_j f(a)$ exists $(1 \leq j \leq n)$
(c) f is differentiable at a
(d) None

(vi) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x) = \sqrt{|x_1x_2|}$. Then

(a) f is continuous only at origin
(b) f is continuous
(c) f is differentiable
(d) f is differentiable only at origin

(vii) Let S and T be k-tensors on V. Then

(a) $S \otimes T = T \otimes S$
(b) $S - T = T - S$
(c) $S + T = T + S$
(d) none

(viii) The dimension of $\mathcal{T}^4(\mathbb{R}^3)$ is

(a) 81 (b) 12 (c) 64 (d) 7

Q.2 Attempt any seven. [14]

(i) Define Euclidean norm and inner product on $\mathbb{R}^n$.
(ii) Prove that $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in \mathbb{R}^n)$.
(iii) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be linear. Prove that T is continuous.
(iv) Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as $f(x) = x^2 + 3x$. Prove that $Df(5) = \lambda_{13}$.
(v) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at a, then prove that each f^i is differentiable at a.
(vi) Define the differentiability of $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at $a \in \mathbb{R}^n$.
(vii) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $a, x \in \mathbb{R}^n$. Show that $D_{sx} f(a) = sD_x f(a) (s \in \mathbb{R})$.
(viii) Let $T : (\mathbb{R}^3)^2 \longrightarrow \mathbb{R}$ be $T(x, y) = x_1 + y_2$. Does $T \in \mathcal{T}^2(\mathbb{R}^3)$? Why?
(ix) Define tensor product and wedge product.

(P.T.O.)

Q.3

(a) Let $x, y \in \mathbb{R}^n$. Prove that $|\langle x, y\rangle| = \|x\|\|y\|$ iff x and y are linearly dependent. [6]

(b) Let $A \subset \mathbb{R}^n$ be closed, let $f : A \longrightarrow \mathbb{R}$ be a bounded function, and let $\varepsilon > 0$. Then prove that the set $B = \{x \in A : o(f;x) \geq \varepsilon\}$ is closed in $\mathbb{R}^n$. [6]

OR

(b) Define $T(x) = (x_1 + x_2, 2x_1 + x_2)$ $(x \in \mathbb{R}^2)$. Find a matrix A such that $T(x) = xA$ $(x \in \mathbb{R}^2)$. [6]

Q.4

(a) If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then there exists a unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that [6]

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

(b) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$. Then prove that fg is differentiable at a. [6]

OR

(b) State and prove the chain rule. [6]

Q.5

(a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$. Then $D_j f^i(a)$ exists for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. Moreover, the Jacobian matrix [6]

$$f'(a) = \begin{bmatrix} D_1 f^1(a) & D_2 f^1(a) & \cdots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \cdots & D_n f^2(a) \\ \vdots & \vdots & \cdots & \vdots \\ D_1 f^m(a) & D_2 f^m(a) & \cdots & D_n f^m(a) \end{bmatrix}$$

(b) Prove that every continuously differentiable function is differentiable. [6]

OR

(b) Find the derivation of $f(x) = (x_1, \cos(x_2 x_3), x_2)$ at $a = (0, 1, \pi)$. [6]

Q.6

(a) Let V be a vector space with dimension n and $k \in \mathbb{N}$. Prove that $\dim(\mathcal{T}^k(V)) = n^k$. [6]

(b) Let $S \in \mathcal{T}^k(V)$ such that $\text{Alt}(S) = 0$ and $T \in \mathcal{T}^\ell(V)$. Prove that $\text{Alt}(S \otimes T) = 0$. [6]

OR

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable. Then prove that [6]

$$\widetilde{f}_{1*}(d\pi_i) = \sum_{j=1}^{n} D_j f^i \cdot d\pi_j \ \ (1 \leq i \leq m).$$

—— X ——