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SEAT No.________

[42]

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Sardar Patel University
Mathematics
M.Sc. Semester I (NC)
Tuesday, 10 April 2018
10.00 a.m. to 1.00 p.m.
PS01CMTH21 - Complex Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) Let $z \in \mathbb{C}$. Which of the following is true?

(a) $|\text{Re } z| + |\text{Im } z| \le |z|$ (c) $|\text{Re } z| + |\text{Im } z| = |z|$
(b) $|\text{Re } z| + |\text{Im } z| \ge |z|$ (d) $|\text{Re } z| \ge |z|$

(2) If $\lim_{z\to z_0} |f(z)| = |w_0|$ implies that $\lim_{z\to z_0} f(z) = w_0$, then

(a) $w_0 = 0$ (b) $|w_0| = 1$ (c) $|w_0| \ge 0$ (d) $w_0 \in \mathbb{R}$

(3) Let v and V be harmonic conjugates of u on a domain D. Which of the following is not true?

(a) $V - v$ is a constant map (c) $v_y - V_y = 0$
(b) $(v - V)_x = 0$ (d) $v = V$

(4) The set of zeros of $\sinh z$ is

(a) $\{n\pi : n \in \mathbb{Z}\}$ (b) $\{n\pi i : n \in \mathbb{Z}\}$ (c) $\{\frac{2n+1}{2}\pi : n \in \mathbb{Z}\}$ (d) $\{\frac{2n+1}{2}\pi i : n \in \mathbb{Z}\}$

(5) The value of $\int_{|z|=1} e^{-z^2} dz$ is ____

(a) 0 (b) $\frac{\sqrt{\pi}}{2}$ (c) $\sqrt{\pi}$ (d) π

(6) Which of the following is a bounded function?

(a) $\sin^2 z + \cos z$ (b) $\cosh^2 z - 1$ (c) $\sinh^2 z + \cosh^2 z$ (d) None of these

(7) The series $\sum_{n=0}^{\infty} nz^n$ converges for ____

(a) $|z| = 1$ (b) $|z| \le 1$ (c) $|z| \ge 1$ (d) $|z| < 1$

(8) 0 is ____ of $z \sin \frac{1}{z}$.

(a) a removable singularity (c) a pole of order 2
(b) a pole of order 1 (d) an essential singularity

Q.2 Attempt any ***Seven***. [14]

(a) If $\lim_{z\to z_0} f(z) = w_0$ and $w_0 \neq 0$, then show that there is $\delta > 0$ and $c > 0$ such that $|f(z)| \ge c$ whenever $0 < |z - z_0| < \delta$.

(b) Find the product of all roots of $z^{20} = 1$.

(P.T.O.)

(c) Is the set $\{z \in \mathbb{C} : |z| \geq 1\}$ domain? Why?
(d) Show that the sum of two harmonic functions is a harmonic function.
(e) Find real and imaginary parts of $f(z) = ze^z$.
(f) If C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction, then show that $\left|\int_C (e^z - \bar{z})dz\right| \leq 60$.
(g) If the real part of entire function f is bounded above, then show that f is a constant map.
(h) Find the Laurent series expansion of $\frac{1}{(z-1)(z-3)}$ in $1 < |z| < 3$.
(i) Evaluate $\int_{|z|=1} \frac{\sin z}{z} dz$.

Q.3

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be $f(0) = 0$ and $f(z) = \frac{\bar{z}^2}{z}$ if $z \neq 0$. Is the function f differentiable at 0? Are the Cauchy-Riemann equations satisfied at $(0,0)$? Justify. [6]

(b) If z and w are nonzero complex numbers, then show that $\arg(zw) = \arg z + \arg w$. Also, prove that $\arg \bar{z} = \arg z^{-1} = -\arg z$. [6]

OR

(b) Let $f = u + iv$ be defined on D, and let $z_0 = x_0 + iy_0 \in D$. Show that f is continuous at z_0 if and only if both u and v are continuous at (x_0, y_0). [6]

Q.4

(c) Let $N(z_0, R)$ be the disc of convergence of the power series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. If C is a contour lying in $N(z_0, R)$ and if g is a continuous function on C, then show that $\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$. [6]

(d) Let f be analytic on a domain D. If $f'(z) = 0$ for all $z \in D$, then show that f is a constant map. [6]

OR

(d) Let $u : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ be $u(x,y) = y/(x^2 + y^2)$. Show that u is harmonic. Find an analytic function on $\mathbb{C} - \{0\}$ whose real part is u. [6]

Q.5

(e) If P is a polynomial of degree $n \geq 1$, then show that there is $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. [6]

(f) If a function f is analytic and not constant in on a domain D, then show that $|f|$ has no maximum value in D. [6]

OR

(f) If $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points 0, 1, $1 + i$ and i, the orientation of C being in the counterclockwise direction, then evaluate $\int_C f(z)dz$. [6]

Q.6

(g) Let z_0 be an isolated singularity of f. Prove that z_0 is a pole of f order m if and only if there is a function φ which is analytic at z_0, $\varphi(z_0) \neq 0$ and $f(z) = \frac{1}{(z-z_0)^m}\varphi(z)$ for all z in some deleted neighborhood of z_0. [6]

(h) Evaluate $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$. [6]

OR

(h) If f is analytic on the open disc $N(z_0, R)$, then show that f has the power series representation $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ for all $z \in N(z_0, R)$. [6]