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SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - I Examination (NC)
Thursday, 19^{th} April, 2018
PS01CMTH04, Linear Algebra

Time: 10:00 a.m. to 01:00 p.m. **Maximum marks: 70**

Note: 1. Figures to the right indicate marks of the respective question.
2. Assume standard notations wherever applicable.

Q-1 Write the question number and appropriate option number only for each question. [8]

1. Let V be a vector space with $\dim V = 4$. Then $\dim \hat{V} =$ ______.
   (a) 2 (b) 4 (c) 8 (d) 16
2. Dimension of the vector space $\mathbb{C}^2$ over the field $\mathbb{C}$ is ______.
   (a) 1 (b) 2 (c) 4 (d) infinite
3. Let V be a finite dimensional vector space over F. If $T \in A(V)$ is one-one, then T is ______.
   (a) singular (b) onto (c) I (d) nilpotent
4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $T(x, y) = (y, -x)$. Then minimal polynomial for T is ______.
   (a) x^2 (b) $1 + x^2$ (c) $1 - x^2$ (d) $1 + x + x^2$
5. Let $V = \mathbb{R}^3$ and $T \in A(V)$. If $T = 0$, then the invariants of T are ______.
   (a) 3 (b) 2, 1 (c) 3, 1 (d) 1, 1, 1
6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x_1, x_2, x_3) = (0, x_1, x_2), \forall (x_1, x_2, x_3) \in \mathbb{R}^3$. Then T is ______.
   (a) one-one (b) onto (c) nilpotent (d) regular
7. Let $A \in M_n(F)$ be nilpotent. Then $\det(A) =$ ______.
   (a) 1 (b) 0 (c) $\neq 0$ (d) n
8. Let $A \in M_n(\mathbb{C})$ with $\det(A) = -1$. Then $\det(A^{-1}) =$ ______.
   (a) -1 (b) 1 (c) i (d) $-i$

Q-2 Attempt *Any Seven* of the following: [14]

(a) Define internal direct sum of vector spaces.

(b) For subspaces V_1 and V_2 of a vector space V over F, show that $V_1 \cup V_2$ need not be a subspace of V.

(c) Let V be a finite-dimensional vector space over F and $S, T \in A(V)$. Show that $r(TS) \leq r(T)$.

(d) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (y - 2z, z - 2x, x - 2y)$, $(x, y, z) \in \mathbb{R}^3$. Find the matrix of T with respect to standard basis of $\mathbb{R}^3$.

(e) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (y, 0, 0)$. Show that T is nilpotent and hence find its invariants

(f) Let V be a vector space over F and $S, T \in A(V)$ be nilpotent. Show that $S + T$ is nilpotent.

(g) Let F be a field and $A \in M_n(F)$ be regular. Then show that $\det(A) \neq 0$.

(h) Find the symmetric matrix associated to the following quadratic form:
$-y^2 - 2z^2 + 4xy + 8xz - 14yz$.

(i) For $A, B \in M_n(\mathbb{R})$, show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Q-3 (a) Let V and W be vector spaces over F of dimensions m and n respectively. Prove that $\dim \operatorname{Hom}(V, W) = mn$ over F. [6]

(P.T.O.)

(b) Let V be a finite-dimensional vector space over a field F and W be a subspace of V. Show that W is also finite-dimensional and in fact $\dim V/W = \dim V - \dim W$. [6]

OR

(b) Let V be a finite-dimensional vector space over F and W be a subspace of V. Show that $\dim W^0 = \dim V - \dim W$, where W^0 is the annihilator of W. [6]

Q-4 (a) Let V be a vector space over F and $T \in A(V)$. If $\lambda_1, \lambda_2, \ldots, \lambda_k \in F$ are distinct characteristic roots of T and $v_1, v_2, \ldots, v_k$ are characteristic vectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_k$ respectively, then show that $v_1, v_2, \ldots, v_k$ are linearly independent. [6]

(b) Let V be a vector space over F and $T \in A(V)$. Show that T is invertible if and only if the constant term of the minimal polynomial for T is non-zero. [6]

OR

(b) Let V be a finite-dimensional vector space over F and $T \in A(V)$. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{D} = \{w_1, w_2, \ldots, w_n\}$ be two bases of V over F. If B and D are matrices of T with respect to the bases $\mathcal{B}$ and $\mathcal{D}$ respectively, then show that B and D are similar matrices. [6]

Q-5 (a) Let V be an n-dimensional vector space over F and $T \in A(V)$ be such that all its characteristic roots are in F. Prove that T satisfies a polynomial of degree n over F. [6]

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Prove that the invariants of T are unique. [6]

OR

(b) Let V be a finite dimensional vector space over F, $T \in A(V)$ be nilpotent such that $T^k = 0$ but $T^{k-1} \neq 0$. Let $v \in V$ such that $T^{k-1}v \neq 0$. Show that $v, Tv, \ldots, T^{k-1}v$ are linearly independent and that their span is a subspace of V invariant under T. [6]

Q-6 (a) For $A, B \in M_n(F)$, show that $\det(AB) = \det(A)\det(B)$. [6]

(b) i. State and prove Jacobson lemma. [4]

ii. For $A, B \in M_2(\mathbb{R})$, by giving an example, show that $\det(A+B) \neq \det(A) + \det(B)$. [2]

OR

(b) Let F be a field of characteristic 0, V be a vector space over F and $T \in A(V)$. If $\operatorname{tr}(T^i) = 0$ for all $i \geq 1$ then show that T is nilpotent. [6]

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