[38]

SEAT No.________

No. of Printed Pages : 3

Sardar Patel University

M.Sc.(Mathematics)(Sem-I)(CBCS) Examination 2018; (NC)

PS01CMTH03 : Functions of Several Real Variables;

12/04/2018; Thursday; Time: 10.00 am to 01:00 pm

Maximum Marks 70

Note: All notations and terminologies are standard;

Que.1: Choose correct answer from the given choices. *(08)*

1. Let $x = (1, -1, 2), y = (-1, 1, -2) \in \mathbb{R}^3$. Then $\langle x, y\rangle =$
   (a) -2; (b) -4; (c) -6; (d) -8;

(i) Let $x, y \in \mathbb{R}^n$. Then which of the following is true?
   (a) $||x|| = \sum_{i=1}^{n} |x_i|$ (b) $||x|| = \sum_{i=1}^{n} |x_i|^2$;
   (c) $||x|| \leq \sum_{i=1}^{n} |x_i|$ (d) $||x|| \leq \sum_{i=1}^{n} |x_i|^2$;

(ii) Let $x, y \in \mathbb{R}^n$ be orthonormal vectors. Then
   (a) $\|x + y\| = \|x - y\|$ (b) $\|x + y\|^3 = \|x - y\|^3$;
   (c) $\|x + y\|^2 = \|x - y\|^2$ (d) None.

(iii) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map.
   (a) If T is angle preserving, then T is norm preserving;
   (b) If T is norm preserving, then T is inner product preserving;
   (c) Both (a) and (b) are true; (d) None.

(iv) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_x f(a)$ exists for all $x \in \mathbb{R}^n$. Then
   (a) $D_j f(a)$ exists $(1 \leq j \leq n)$ (b) f is continuous at a;
   (c) f is differentiable at a (d) None;

(v) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_j f(a)$ exists for all $1 \leq j \leq n$. Then
   (a) f is continuous at a (b) f is continuously differentiable at a;
   (c) $D_x f(a)$ exists (d) None of these;

(vi) Which of the following is a 3-tensor on $\mathbb{R}^5$?
   (a) $T(x, y, z) = x_1 y_2$ (b) $T(x, y, z) = x_1 y_2 z_3$;
   (c) $T(x, y, z) = y_1 z_2$ (d) $T(x, y, z) = x_1 y_2 z_3$;

(vii) The dimension of $\Lambda^4(\mathbb{R}^6)$ is
   (a) 1296; (b) 4096; (c) 15; (d) 0;

(P.T.O.)

(viii) Let $\omega \in \Lambda^2(V)$ and $\eta \in \Lambda^3(V)$. Then
(a) $\omega \otimes \eta = \eta \otimes \omega$; (b) $\omega \wedge \eta = \eta \wedge \omega$;
(c) $\omega \wedge \eta = -\eta \wedge \omega$; (d) All are true;

Que.2: Attend any seven. *(14)*

(i) Let $x, y \in \mathbb{R}^n$. Prove that $|\langle x, y\rangle| \leq \|x\|\|y\|$.

(ii) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous. Prove that $f + g$ is continuous.

(iii) Define derivation of $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at a. Give an example of a function which is continuous but not differentiable at a.

(iv) Define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as $f(x) = |x_1x_2|$. Prove that f is differntiable at 0. What is $Df(0)$?

(v) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as $f(x) = e^{x_1}$. Find $Df(0)$.

(vi) Let $A \subset \mathbb{R}^n$ be open, $a \in A$, and $f : A \longrightarrow \mathbb{R}$. If f has maximum value at the point a and $D_if(a)$ exists, then show that $D_if(a) = 0$.

(vii) Define directional derivative. Is it true that if directional derivatives exist, then partial derivatives also exist?

(viii) Define alternating k-tensor. Let $T(x, y) = \frac{1}{2}(x_1y_2 - x_2y_1)$ $(x, y \in \mathbb{R}^2)$. Does $T \in \Lambda^2(\mathbb{R}^2)$?

(ix) Define Fields and Forms.

Que.3: (A) State and prove the Polarization Identity. *(06)*
(B) Let $A \subset \mathbb{R}^n$ be closed, $f : A \longrightarrow \mathbb{R}$ be bounded, and $\varepsilon > 0$. Then prove that the set $B = \{x \in A : o(f; x) \geq \varepsilon\}$ is closed in $\mathbb{R}^n$. *(06)*

OR

(B) Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. Give the definition of $\|T\|$ and prove that $\|T(x)\| \leq \|T\|\|x\|$ $(x \in \mathbb{R}^n)$. *(06)*

Que.4: (A) If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then prove that there exists a unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that *(06)*

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

(B) State and prove the chain rule. *(06)*

OR

(B) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x) = \frac{|x_1 x_2|}{||x||}$ if $x \neq 0$ and $f(0) = 0$. Is the function f differentiable at 0? Justify your answer. *(06)*

Que.5: (A) Find $Df(a)$ using Jacobian matrix, where $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined as $f(x) = (\cos(x_1 x_2), x_1^5 + 3x_2)$ and $a = (0, \pi/2)$. *(06)*
(B) Prove that continuously differentiable function is differentiable. *(06)*

OR

(B) Give a detailed example of a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $D_x f(0)$ exists for all $x \in \mathbb{R}^2$ but f is not continuous at 0. *(06)*

Que.6: (A) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Let $\widetilde{f_{k*}} : \Delta_{kF}(\mathbb{R}^m) \to \Delta_{kF}(\mathbb{R}^n)$ be defined as $\widetilde{f_{k*}}(\omega)(p) := \widetilde{f_{pk}^*}(\omega(f(p)))$ $(p \in \mathbb{R}^n; \omega \in \Delta_{kF}(\mathbb{R}^m))$. Then prove that $\widetilde{f_{k*}}$ is well-defined and it is a linear map. *(06)*
(B) Define the wedge product. Prove that it is associative. Explicitly state the results used in the proof. *(06)*

OR

(B) Define $\text{Alt}(T)$. Prove that if $\omega \in \Lambda^k(V)$, then $\text{Alt}(\omega) = \omega$. *(06)*

******——X——******