SEAT No.________

[55]

No of printed pages: **2**

Sardar Patel University

Mathematics

M.Sc. Semester I

Tuesday, 11 April 2017

10.00 a.m. to 1.00 p.m.

PS01CMTH01 - Complex Analysis I

Maximum Marks: 70

Q.1 Fill in the blanks. [8]

(1) The principal argument of $1+i$ is _____

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

(2) Let $z \in \mathbb{C}$. Then $z = \bar{z}$ if and only if _____

(a) z is real (b) z is imaginary (c) $|z| = 1$ (d) $z = 0$

(3) Which of the following are Cauchy-Riemann equations in polar coordinates?

(a) $ru_r = v_\theta, rv_r = -u_\theta$ (c) $ru_\theta = v_r, rv_\theta = -u_r$
(b) $ru_r = -v_\theta, rv_r = u_\theta$ (d) $ru_\theta = -v_r, rv_\theta = u_r$

(4) Which of the following is not an entire function?

(a) $\frac{1}{e^z}$ (b) $\frac{1}{e^{z^2}}$ (c) $\frac{1}{e^{z^2+z}}$ (d) $\frac{1}{z^2+1}$

(5) The value of $\int_{|z|=1} \frac{\sinh z}{z^4} dz$ is _____

(a) 0 (b) $\frac{\pi i}{3}$ (c) $\frac{3}{\pi i}$ (d) $\frac{5!}{2\pi i}$

(6) Which of the following is a bounded function?

(a) $\sin z + \cos z$ (b) $\cosh z$ (c) $\sinh z$ (d) None of these

(7) The series $\sum_{n=0}^{\infty} z^n$ converges for _____

(a) $|z| = 1$ (b) $|z| \leq 1$ (c) $|z| \geq 1$ (d) $|z| < 1$

(8) 0 is _____ of $e^{\frac{1}{z}}$.

(a) a removable singularity (c) a pole of order 2
(b) a pole of order 1 (d) an essential singularity

Q.2 Attempt any *Seven*. [14]

(a) If $a, b \in \mathbb{C}$, then show that $||a| - |b|| \leq |a - b|$.
(b) If $z \neq 0$, then show that $\arg(z^{-1}) = -\arg(z)$.
(c) Is the set $\{z \in \mathbb{C} : |z| \leq 1\}$ domain? Why?
(d) Show that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

(e) Find real and imaginary parts of $f(z) = e^{-z^3}$.

(f) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that $\left|\int_C \frac{z+4}{z^3-1}dz\right| \leq \frac{6\pi}{7}$.

(g) State Principle of deformation of paths.

(h) Find the Laurent series expansion of $\frac{1}{(z-2)(z-3)}$ in $2 < |z| < 3$.

(i) Find the fixed points of the bilinear transformation $w = \frac{z-1}{z+1}$.

Q.3

(a) If $z_1, z_2, \ldots, z_n$ are complex numbers, then show that $|\sum_{i=1}^{n} z_i| \leq \sum_{i=1}^{n} |z_i|$. [6]

(b) Define n th root of a complex number. Find all $z \in \mathbb{C}$ such that $z^4 = -16$. [6]

OR

(b) (1) If $z \in \mathbb{C}$, then show that $|\text{Re } z| + |\text{Im } z| \leq \sqrt{2}|z|$. [3]

(2) Use de Moivre's formula to derive $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$. [3]

Q.4

(c) Suppose that $f = u + iv$ be defined in an ϵ- neighborhood of $z_0 = x_0 + iy_0$. If u_x, u_y, v_x and v_y exist and continuous in a neighborhood of (x_0, y_0) and if Cauchy-Riemann equations are satisfied at (x_0, y_0), then show that f is differentiable at z_0. [6]

(d) Let f be analytic on a domain D. If either $\overline{f}$ is analytic on D or if $|f|$ is constant on D, then show that f is a constant map. [6]

OR

(d) Define a harmonic function. Construct an analytic function whose imaginary part is $v(x, y) = e^{2x}\sin 2y - y$. [6]

Q.5

(e) Let f be analytic within and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then show that $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$. Hence evaluate $\int_{|z|=1} \frac{\cosh^2 z}{z}dz$. [6]

(f) State Liouville's Theorem. Deduce Fundamental Theorem of Algebra from it. [6]

OR

(f) If $w : [a, b] \to \mathbb{C}$ is integrable, then show that $\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$. Also, define $\int_C f(z)dz$. [6]

Q.6

(g) State Taylor's Theorem. Hence find the Taylor series expansion of $\sin z$ and $\cosh z$ about 0 in appropriate regions. [6]

(h) Evaluate $\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}dx$. State the result you use. [6]

OR

(h) When is an isolated singularity z_0 of f called a pole? What is meant by the order of a pole? Show that 0 is a pole of $\frac{1-\cos z}{z^3}$. What is the order of this pole? [6]

ЩЩЩЩЩЩЩЩЩ

SEAT No.________

[443]

No of printed pages: 2

Sardar Patel University

M.Sc. Semester I Examination

Monday, 17^{th} April 2017; 10.00 to 13.00

Subject: Mathematics; Code: PS01CMTH02; Title of Paper: Topology-I

Maximum Marks: 70

Q.1 For each question, write the correct **option number only**. [8]

(*a*) ____ is a closed subset of $\mathbb{R}$

(i) $\mathbb{N}$ (ii) $\mathbb{Q}$ (iii) $(0,1]$ (iv) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(*b*) Boundary of ____ $\subset \mathbb{R}$ is connected.

(i) $\{0.98765\}$ (ii) $\mathbb{Q}$ (iii) $[0,1]$ (iv) $(0,1)$

(*c*) Which of the following guarantees the continuity of a function $f : X \to Y$?

(i) X is discrete.
(ii) Y is discrete.
(iii) $f(X)$ is compact.
(iv) f is one-one and onto.

(*d*) A metric space is ____.

(i) connected (ii) Hausdorff (iii) compact (iv) separable

(*e*) ____ is a compact subset of $\mathbb{R}^2$.

(i) $\mathbb{Q}$ (ii) $\mathbb{N} \cap (0, 25.3)$ (iii) $(0,1)$ (iv) $(0,1]$

(*f*) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) =$ ____ is not uniformly continuous.

(i) x^2 (ii) $|x|$ (iii) x (iv) $\sin x$

(*g*) A finite subset of a ____ space need not be closed.

(i) T_1 (ii) cocountable (iii) cofinite (iv) indiscrete

(*h*) A ____ subspace of normal space is normal.

(i) bounded (ii) complete (iii) connected (iv) closed

Q.2 Attempt any **Seven. (Start a new page.)** [14]

(*a*) Show that there is only one Hausdorff topology on a finite set.
(*b*) Define and give example of a T_1-*space*.
(*c*) Show that every metric space is T_2.
(*d*) For a subset A of a metric space X, show that the function $f(x) = d(x, A)$, $(x \in X)$, is continuous.
(*e*) State Cantor's intersection theorem.
(*f*) Show that a compact metric space is bounded.
(*g*) Prove that $\mathbb{R}$ with cocountable topology is not compact.
(*h*) Show that a continuous function from a connected space to a discrete space is constant.
(*i*) Define a *second countable space*. Show that $\mathbb{R}$ with the usual topology is second countable.

[Contd...]

Q.3 (Start a new page.)

(*a*) Define *cocountable topology* on a set X. Show that it is a topology. [6]

(*b*) (i) Define *interior of a set*. In a topological space X, show that $A \subset X$ is open if $A \cap \text{bd}(A) = \emptyset$. [3]

(ii) Define *limit point of a set*. Find all limit points of $\mathbb{Q}$ in $\mathbb{R}$ with cocountable topology. [3]

OR

(*b*) (i) Show that arbitrary intersection of closed sets is closed. [3]

(ii) Give an example to show that arbitrary intersection of open sets need not be open. [3]

Q.4 (Start a new page.)

(*c*) Let $X = \prod_{i=1}^{n} X_i$ be a product of topological spaces. Show that a X is T_2 if and only if each X_i is a T_2-space. [6]

(*d*) Show that in a metric space, every convergent sequence is Cauchy. Also show that a uniformly continuous function maps a Cauchy sequence to Cauchy sequence. [6]

OR

(*d*) Show that projections are open and continuous. [6]

Q.5 (Start a new page.)

(*e*) Show that $[0, 1]$ is compact. [6]

(*f*) (i) Show that a continuous image of a compact space is compact. [3]

(ii) Show that a closed subspace of a compact space is compact. [3]

OR

(*f*) Show that a totally bounded subset of a metric space is bounded. Also show that a compact subset of a metric space is totally bounded. [6]

Q.6 (Start a new page.)

(*g*) Define *a normal space* and show that a a normal space is regular. [6]

(*h*) Show that a finite product of T_3-spaces is T_3. [6]

OR

(*h*) Show that $[0, 1]$ is a connected subset of $\mathbb{R}$. [6]

✠‡✠‡✠‡✠†✠

[61] Seat No. ____________ No. of printed pages : 2

SARDAR PATEL UNIVERSITY
M.Sc. (Mathematics) (External) Examination
Thursday, 13th April 2017
10.00 am - 1.00 pm
PS01CMTH03 : Functions of Several Real Variables

Total Marks: 70

Note: Notations and Terminologies are standard.

Q.1 Choose correct option from given four choices. [08]

(i) Let $x = (\sqrt{2}, 1, e)$ and $y = (\sqrt{2}, 2, e)$. Then $\|x - y\| =$

(a) $\|x\|$ (b) $\|y\|$ (c) $\|x\|\|y\|$ (d) none

(ii) Which of the following is true?

(a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 0$ (b) $\lim\limits_{x\to 0} \frac{\cos x}{x} = 1$ (c) $\lim\limits_{x\to 0} x\cos(\frac{1}{x}) = 0$ (d) none

(iii) Let $a = (2, 1)$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x) = x_1x_2$. Then $Df(a) =$

(a) $\pi_1 + \pi_2$ (b) $2\pi_1 + \pi_2$ (c) $\pi_1 + 2\pi_2$ (d) none

(iv) Let $a \in \mathbb{R}^n$ be fixed. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous at a. Then

(a) $Df(a)$ exists (b) $D_xf(a)$ exist (c) $D_jf(a)$ exist (d) none

(v) Let $a \in \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $D_xf(a)$ exists for all $x \in \mathbb{R}^n$. Then

(a) $D_jf(a)$ exists for all $1 \le j \le n$
(b) $Df(a)$ exists
(c) f is continuous at a
(d) f is continuously differentiable at a

(vi) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x) = \sqrt{|x_1x_2|}$. Then

(a) f is continuous only at origin
(b) f is continuous
(c) f is differentiable
(d) f is differentiable only at origin

(vii) Let S and T be k-tensors on V. Then

(a) $S \otimes T = T \otimes S$
(b) $S - T = T - S$
(c) $S + T = T + S$
(d) none

(viii) The dimension of $\mathcal{T}^6(\mathbb{R}^4)$ is

(a) 10 (b) 15 (c) 24 (d) 1296

Q.2 Attempt any **seven.** [14]

(i) Prove that $\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2]$ $(x, y \in \mathbb{R}^n)$.
(ii) Prove that $\|x + y\| \le \|x\| + \|y\|$ $(x, y \in \mathbb{R}^n)$.
(iii) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Prove that T is norm preserving iff it is inner product preserving.
(iv) Define the differentiability of $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ at $a \in \mathbb{R}^n$.
(v) Prove that every linear map is differentiable and its derivation is itself.
(vi) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has maximum value at a and $D_if(a)$ exists, then show that $D_if(a) = 0$.
(vii) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $a, x \in \mathbb{R}^n$. Show that $D_{sx}f(a) = sD_xf(a)(s \in \mathbb{R})$.
(viii) Let $T : (\mathbb{R}^3)^2 \longrightarrow \mathbb{R}$ be $T(x, y) = x_1 + y_2$. Does $T \in \mathcal{T}^2(\mathbb{R}^3)$? Why?
(ix) Define tensor product and wedge product.

(Continue on Page-2)

Q.3

(a) Let $x, y \in \mathbb{R}^n$. Prove that $|\langle x, y\rangle| = \|x\|\|y\|$ iff x and y are linearly dependent. [6]

(b) Let $A \subset \mathbb{R}^n$ be closed, let $f : A \longrightarrow \mathbb{R}$ be a bounded function, and let $\varepsilon > 0$. Then prove that the set $B = \{x \in A : o(f; x) \geq \varepsilon\}$ is closed in $\mathbb{R}^n$. [6]

OR

(b) Define $T(x) = (x_1 + x_2, 2x_2 + x_3, x_1 - 2x_2 + 3x_3)$ $(x \in \mathbb{R}^3)$. Find 3×3 matrix A such that $T(x) = xA$ $(x \in \mathbb{R}^3)$. [6]

Q.4

(a) If a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then there exists a unique linear transformation $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that [6]

$$\lim_{h \to 0} \frac{||f(a+h) - f(a) - \lambda(h)||}{||h||} = 0.$$

(b) Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable at $a \in \mathbb{R}^n$. Then prove that fg is differentiable at a. [6]

OR

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $a \in \mathbb{R}^n$. Then prove that f is differentiable at a iff each component $f^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(1 \leq i \leq m)$ is differentiable at a. Moreover, in this case, show that $Df(a) = (Df^1(a), \ldots, Df^m(a))$. [6]

Q.5

(a) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$. Then $D_j f^i(a)$ exists for all $1 \leq i \leq m$ and for all $1 \leq j \leq n$. Moreover, the Jacobian matrix [6]

$$f'(a) = \begin{bmatrix} D_1 f^1(a) & D_2 f^1(a) & \cdots & D_n f^1(a) \\ D_1 f^2(a) & D_2 f^2(a) & \cdots & D_n f^2(a) \\ \vdots & \vdots & \cdots & \vdots \\ D_1 f^m(a) & D_2 f^m(a) & \cdots & D_n f^m(a) \end{bmatrix}$$

(b) Prove that every continuously differentiable function is differentiable. [6]

OR

(b) Find the derivation of $f(x) = (x_1, \sin(x_2 + x_3), x_2)$ at $a = (0, 1, \pi)$. [6]

Q.6

(a) Let V be a vector space with dimension n and $k \leq n$. Prove that the dimension of $\Lambda^k(V)$ is $\frac{n!}{k!(n-k)!}$. [6]

(b) Let $S \in \mathcal{T}^k(V)$ such that $\mathrm{Alt}(S) = 0$ and $T \in \mathcal{T}^\ell(V)$. Prove that $\mathrm{Alt}(S \otimes T) = 0$. [6]

OR

(b) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be differentiable. Then prove that [6]

$$\widetilde{f}_{1*}(d\pi_i) = \sum_{j=1}^{n} D_j f^i \cdot d\pi_j \quad (1 \leq i \leq m).$$

THE END

[36] SEAT No.________ No. of printed pages: 2

SARDAR PATEL UNIVERSITY

M.Sc. (Mathematics) Semester - I Examination

Wednesday, 19th April, 2017

PS01CMTH04, Linear Algebra

Time: 10:00 a.m. to 1:00 p.m. **Maximum marks: 70**

Note: All the questions are to be answered in answer book only. Figures to the right indicate full marks of the respective question. Assume standard notations wherever applicable.

Q-1 Fill up the gaps in the following: [8]

1. The dimension of the vector space $\mathbb{C}^2$ over $\mathbb{C}$ is ____.
   (a) 1 (b) 2 (c) 4 (d) infinite
2. Let V_1 and V_2 be two subspaces of a vector space V. Then ____ need not be a subspace of V.
   (a) $V_1 + V_2$ (b) $V_1 \cup V_2$ (c) $L(V_1 \cup V_2)$ (d) $V_1 \cap V_2$
3. Let V be a vector space over a field F and $T \in A(V)$ be such that $T^2 + 5T + I = 0$. Then ____.
   (a) T is regular (b) T is singular (c) $\det(T) = 0$ (d) T is not onto
4. Let V be a vector space over F and $T \in A(V)$ be such that $0 \neq T \neq I$ and $T^2 = T$. Then the characteristic roots of T are ____.
   (a) 0 and 1 (b) 0 and 2 (c) 0 and 0 (d) 1 and 1
5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(x_1, x_2, x_3) = (0, x_1, x_2)$. Then the minimal polynomial of T is ____.
   (a) $p(x) = x$ (b) $p(x) = x^2$ (c) $p(x) = 0$ (d) $p(x) = x^3$
6. Let V vector space over F, W be subspaces of a V and $T \in A(V)$. W is invariant under T if ____.
   (a) T must be one-one (b) $T(W) = V$ (c) $T(W) \subset W$ (d) $W \subset T(W)$
7. Let $A \in M_n(F)$ be nilpotent. Then $\det(A) =$____.
   (a) 1 (b) -1 (c) 0 (d) n
8. Let $A, B \in M_n(F)$ for some field F. Then ____.
   (a) $\text{tr}(\lambda A) = \lambda^n \text{tr}(A)$ (c) $\det(A + B) = \det(A) + \det(B)$
   (b) $\det(\lambda A) = \lambda \det(A)$ (d) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

Q-2 Attempt *Any Seven* of the following: [14]

(a) Check whether $v_1 = (1, 2, 1)$, $v_2 = (0, 1, 1)$ and $v_3 = (0, 0, 1)$ are linearly independent over $\mathbb{R}$ or not?

(b) Let $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y = z\} \subset \mathbb{R}^3$. Show that W is a subspace of $\mathbb{R}^3$. What is the dimension of W?

(c) Let V be a finite dimensional vector space over a field F and $T \in A(V)$. Show that $\text{rank}(ST) \leq \text{rank}(T)$.

(d) Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) = (x - y + z, x - 2y, x - 2z)$, $(x, y, z) \in \mathbb{R}^3$. Find the matrix of T with respect to standard basis of $\mathbb{R}^3$.

(e) Let V be a vector space over a field F and $T \in A(V)$. Show that $\ker(T)$ is invariant under T.

(f) Let V be a vector space over F and $S, T \in A(V)$ such that S is nilpotent. Show that ST is nilpotent if $ST = TS$.

(g) Show that similar matrices have same determinant.

(h) Find the inertia of the quadratic equation $x_1 + x_2 + x_3 = 0$.

(i) For $A, B \in M_n(\mathbb{R})$, show that $\text{tr}(AB) = \text{tr}(BA)$.

Q-3 (a) Let V be a finite-dimensional vector space over a field F and W be a subspace of V. Show that W is finite-dimensional and $\dim V/W = \dim V - \dim W$. [6]

(b) Let V be a vector space and $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. If $\{u_1, \ldots, u_m\}$ in V are linearly independent then $m \leq n$. [6]

OR

(b) Let V be a vector space over F. Show that V is isomorphic to a subspace of $\hat{\hat{V}}$. [6]

Q-4 (a) Let $\mathcal{A}$ be an algebra over F. Show that $\mathcal{A}$ is isomorphic to a subalgebra of $A(V)$ for some vector space V over F. [6]

(b) Let V be a vector space over F and $T \in A(V)$. Show that characteristic vectors corresponding to distinct characteristic roots of T are linearly independent. [6]

OR

(b) Let V be a vector space over F and $T \in A(V)$. Show that T is regular if and only if the constant term of the minimal polynomial for T is non-zero. [6]

Q-5 (a) Let V be a finite dimensional vector space over F and $T \in A(V)$ be nilpotent. Then show that the invariants of T are unique. [6]

(b) Let V be an n-dimensional vector space over F. If $T \in A(V)$ has all its characteristic roots in F, then show that T satisfies a polynomial of degree n in $F[x]$. [6]

OR

(b) Let V be a finite dimensional vector space over F and $T \in A(V)$. Let $V = V_1 \oplus V_2$, where V_1 and V_2 are subspaces of V invariant under T. Let $T_i = T|_{V_i}$ and $p_i(x) \in F[x]$ be the minimal polynomial for T_i, $i = 1, 2$. Show that the least common multiple of $p_1(x)$ and $p_2(x)$ is the minimal polynomial for T. [6]

Q-6 (a) For $A, B \in M_n(F)$, show that $\det(AB) = \det(A)\det(B)$. [6]

(b) i. Let V be a finite-dimensional vector space over F and $S, T \in A(V)$ such that $ST - TS$ commutes with S. Show that $ST - TS$ is nilpotent. [4]

ii. Find the symmetric matrix associated with the quadratic form: $-y^2 - 2z^2 + 4xy + 8xz - 14yz$. [2]

OR

(b) Prove that the determinant of a triangular matrix is the product of its entries on the main diagonal. [6]

♯♯♯♯♯♯♯♯♯♯♯♯♯♯♯♯♯♯♯♯

[8] SEAT No.________

No of printed pages: 2

Sardar Patel University

M.Sc. (Sem-I), PS01CMTH05, Methods of Differential Equations;
(Maths) Friday, 21^{st} April, 2017; 10.00 a.m. to 01.00 p.m.

Maximum Marks: 70

Note: (i) Notations and terminologies are standard; (ii) Figures to the right indicate marks.

Q.1 Answer the following. [8]

1. The function $f : \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{1-\cos x}{x^2}, x \neq 0$ and $f(0) = \frac{1}{2}$ is
   (A) not continuous at 0 (B) continuous but not differentiable at 0
   (C) not analytic at 0 (D) None of these
2. The set of singular points of $xy'' - e^x y = 0$ is
   (A) $\mathbb{R}$ (B) $\{0\}$ (C) φ (D) none of these
3. $J_5(x) =$
   (A) $-J_{-5}(x)$ (B) $-J_{-5}(-x)$ (C) $J_5(-x)$ (D) None of these
4. If $\frac{x-1}{2} = c_0P_0(x) + c_1P_1(x)$, then
   (A) $\{c_0, c_1\} \subset \mathbb{Z}$ (B) $c_0 = c_1$ (C) $c_1 \in \mathbb{N}$ (D) None of these
5. Which of the following is an integrating factor of $xdy - ydx$?
   (A) $\dfrac{1}{x}$ (B) $\dfrac{1}{y^2}$ (C) $\dfrac{1}{y}$ (D) none of these
6. The necessary and sufficient condition for integrability of Pfaffian differential equation in three variables is
   (A) $X \cdot \text{curl}X = 0$ (B) $X \cdot \text{curl}X = 1$
   (C) $X \cdot \text{curl}X^2 = 0$ (D) none of these
7. $F(1, \frac{1}{2}; \frac{1}{2}; \frac{1}{3}) =$
   (A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) none of these
8. $F(\alpha, \beta; \gamma; 1)$ converges if
   (A) $\gamma < \alpha + \beta - 1$ (B) $\gamma > \alpha + \beta$ (C) $\gamma > \alpha - \beta$
   (D) $2\alpha < 2\gamma - 1$

Q.2 Attempt any ***seven***: [14]

(a) State Frobenius theorem.

(b) Find the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-1)^n}{(n+1)3^n}$.

(c) Are e^x and $\cos x$ $(x \in \mathbb{R})$ linearly independent? Justify.

(d) Show that between any two positive zeros of J_1 there is a zero of J_0.

(e) Show that $P_n(x) = (-1)^n P_n(-x)$, $n \in \mathbb{N} \cup \{0\}$.

(f) Find a partial differential equation by eliminating F from $F(x^2 + y^2) + xy = z$.

(g) Solve $yzp + xzq = xy$.

(h) State Kummer's formula.

(i) Show that $F(\alpha, \beta; \beta; x) = (1-x)^{-\alpha}$.

Q.3

(a) Solve: $x^2y'' - xy' - (x-1)y = 0$ near 0. [6]

(b) Evaluate $\Gamma(\frac{1}{2})$. [6]

OR

(b) Prove or disprove: the function $f : (-1,1) \to \mathbb{R}$, $f(x) = \frac{1}{x-1}$ is analytic at 0.

Q.4

(a) Prove: $2\alpha J_\alpha(x) = x[J_{\alpha-1}(x) + J_{\alpha+1}(x)]$ [6]

(b) Find the first four terms of the Fourier-Legendre expansion of the function [6]

$$f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1. \end{cases}$$

OR

(b) State Rodrigue's formula and hence find $P_i(x)$, where $i = 0, 1, 2, 3$.

Q.5

(a) Show that $X \cdot \text{curl} X = 0$ iff $\mu X \cdot \text{curl}(\mu X) = 0$ where $X = (P, Q, R)$ and $P, Q, R, \mu (\neq 0)$ are functions of x, y and z. [6]

(b) Solve $y' - 2(x + xy) = 0$, $y(0) = 1$ using Picard's method of successive approximations. [6]

OR

(b) Verify that the differential equation $yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0$ is integrable and find its primitive.

Q.6

(a) State and prove Integral representation of Gauss's hypergeometric function. [6]

(b) Find a complete integral of $(p^2 + q^2)y = qz$ using Charpit's method. [6]

OR

(b) Find a complete integral of $z^2 = pqxy$ using Jacobi's method.

*****

[A-21]

SEAT No.________

No of printed pages: 2

Sardar Patel University

M.Sc. Semester I Examination

Monday, 17^{th} April 2017; 10.00 to 13.00

Subject: Mathematics; Code: PS01CMTH07; Title of Paper: Topology-I

Maximum Marks: 70

Q.1 For each question, write the correct **option number only**. [8]

(*a*) _____ is a open subset of $\mathbb{R}$

(i) $\mathbb{N}$ (ii) $\mathbb{Q}$ (iii) $(0,1)$ (iv) $\{\frac{1}{n} : n \in \mathbb{N}\}$

(*b*) Boundary of _____ $\subset \mathbb{R}$ is compact.

(i) $\mathbb{N}$ (ii) $\mathbb{Q}$ (iii) $\mathbb{Z}$ (iv) $(0,1)$

(*c*) A bijective continuous function $f : [0,1] \to Y$ is a homeomorphism if _____.

(i) Y is compact (iii) $f(X)$ is compact
(ii) Y is T_2 (iv) f is constant

(*d*) A metric space is _____.

(i) complete (ii) T_4 (iii) compact (iv) separable

(*e*) _____ is a compact subset of $\mathbb{R}^2$.

(i) $\mathbb{Q}$ (ii) $\mathbb{N} \cap [0, 25.3]$ (iii) $(0,1)$ (iv) $(0,1]$

(*f*) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) =$ _____ is not uniformly continuous.

(i) x^2 (ii) $|x|$ (iii) x (iv) $\sin x$

(*g*) A finite subset of a _____ space need not be closed.

(i) T_1 (ii) cocountable (iii) cofinite (iv) indiscrete

(*h*) A _____ subspace of normal space is normal.

(i) bounded (ii) complete (iii) countable (iv) closed

Q.2 Attempt any **Seven. (Start a new page.)** [14]

(*a*) Give four distinct topologies on $\{a, b, c, d, e\}$.
(*b*) Show that the lower limit topology is finer than the usual topology.
(*c*) Prove that $f(x) = |x - 2|$, $(x \in \mathbb{R})$, is continuous on $\mathbb{R}$.
(*d*) Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$.
(*e*) Show that a finite set is compact with every topology on it.
(*f*) Define and give example of a T_3-*space*.
(*g*) Show that a compact subset of $\mathbb{R}$ is bounded.
(*h*) Prove that $\mathbb{R}$ with cofinite topology is not regular.
(*i*) Define a *second countable space*. Show that $\mathbb{R}$ with the usual topology is second countable.

[Contd...]

Q.3 (Start a new page.)

(a) Define *cofinite topology* on a set X. Show that it is a topology. [6]

(b) (i) Define *boundary of a set.* In a topological space X, show that $A \subset X$ is open if $A \cap \text{bd}(A) = \emptyset$. [3]

(ii) Define *limit point of a set.* Find all limit points of $[0,1]$ in $\mathbb{R}$ with cocountable topology. [3]

OR

(b) (i) Show that arbitrary intersection of closed sets is closed. [3]

(ii) Give an example to show that arbitrary union of closed sets need not be closed. [3]

Q.4 (Start a new page.)

(c) Let $X = \prod_{i=1}^{n} X_i$ be a product of topological spaces. Prove that X_1 is homeomorphic to a subset of X.

(d) Show that in a metric space, every convergent sequence is Cauchy. Also show that a uniformly continuous function maps a Cauchy sequence to Cauchy sequence. [6]

OR

(d) State and prove Cantor's intersection theorem. [6]

Q.5 (Start a new page.)

(e) Show that $[0,1]$ is compact. [6]

(f) (i) Show that a continuous image of a compact space is compact. [3]

(ii) Show that a closed subspace of a compact space is compact. [3]

OR

(f) Show that a finite subset of a metric space is bounded. Also show that a compact subset of a metric space is bounded. [6]

Q.6 (Start a new page.)

(g) Define *a regular space* and show that a a regular space is Hausdorff. [6]

(h) Show that a finite product of T_3-spaces is T_3. [6]

OR

(h) Show that a topological space X is T_4 if and only if for every open subset U and a closed subset F satisfying $F \subset U \subset X$, there exists an open set V such that $F \subset V \subset \overline{V} \subset U$. [6]

✠‡✠‡✠‡✠†✠