[39/A-8]

SARDAR PATEL UNIVERSITY
B.Sc.(SEMESTER - V) EXAMINATION
Thursday , 1^{st} November,2018
MATHEMATICS : US05CMTH05
(Number Theory)

Time : 10:00 a.m. to 01:00 p.m. **Maximum Marks : 70**

Que.1 Fill in the blanks. 10

(1) $(a\ ,\ 1) =$ .... , $\forall\ a \in \mathbb{Z}$

(a) $-a$ (b) $|a|$ (c) a (d) 1

(2) $(a\ ,\ b)[a\ ,\ b] =$ .... $\forall\ a\ ,\ b \in \mathbb{Z}$.

(a) 1 (b) ab (c) $|ab|$ (d) $(a\ ,\ b)$

(3) $[25,30] =$ ....

(a) 750 (b) 60 (c) 150 (d) 5

(4) P(60) = ...

(a) 120 (b) 60 (c) 60^6 (d) 60^5

(5) .... is a Mersenne number .

(a) 100 (b) 127 (c) 32 (d) 125

(6) $\mu(6) =$ ....

(a) 1 (b) 0 (c) -1 (d) 2

(7) Prove that every number containing more than three digits can be divided by 8 iff the number formed by .... digits can be divided by 8.

(a) last two (b) last three (c) first two (d) first three

(8) 765432 is not divisible by ....

(a) 7 (b) 3 (c) 4 (d) 9

(9) If $a^d \equiv 1(mod\ m)$ and n is order of a modulo m then ....

(a) $n < d$ (b) d/n (c) d = n (d) n/d

(10) $2x + 4y \equiv 5(mod\ 12)$ has only .... solutions.

(a) 1 (b) 24 (c) 12 (d) 5

[P.T.O.]

Que.2 Answer the following (Any Ten) 20

(1) Prove that if $(a,b) = d$ then $\exists\, x, y \in \mathbb{Z}$ such that $xa + yb = d$.

(2) Prove that $(a,b) = (b, a+kb)$, for $k \in Z$.

(3) Prove that common multiple of a and b is a multiple of their lcm .

(4) Let x be any positive real number and n be any positive integer then prove that among the integers from 1 to x the number of multipliers of n is $\left[\frac{x}{n}\right]$.

(5) Prove that the successive Fibonacci numbers are relatively prime .

(6) Prove that $u_{n+1}^2 = u_n^2 + 3u_{n-1}^2 + 2[u_{n-2}^2 + u_{n-3}^2 + \cdots + u_1]$.

(7) If $a_1 \equiv b_1 (mod\; n)$ then prove that $a_1{}^m \equiv b_1{}^m (mod\; n)$, $\forall\, m \in \mathbb{N}$.

(8) If $ca \equiv cb(mod\; n)$ and $(c,n) = 1$ then prove that $a \equiv b(mod\; n)$.

(9) Prove that Pythagoras equation has no prime solution .

(10) Solve the equation $12x + 15 \equiv 0(mod\; 45)$.

(11) Find $\phi(243) + \phi(81) + \phi(27) + \phi(9) + \phi(3)$.

(12) If $(a,p) = 1$, p is prime , then prove that $a^{p-1} \equiv 1(mod\; p)$.

Que.3 (a) If P_n is n^{th} prime number then prove that $P_n < 2^{2^n}$, $\forall\, n \in \mathbb{N}$. 5

(b) State and prove unique factorization theorem for positive integers . 5

OR

Que.3 (c) Let g be a positive integer greater than 1 then prove that every positive integer a can can be written uniquely in the form $a = c_n g^n + c_{n-1} g^{n-1} + ... + c_1 g + c_0$, where $n \geq 0$, $c_i \in \mathbb{Z}$, $0 \leq c_i < g$, $c_n \neq 0$. 5

(d) State and prove Fundamental theorem of divisibility . 5

Que.4 (a) In usual notation prove that $P(n!) = \sum_{k=1}^{m} \left[\frac{n}{p^k}\right]$, where $p^m \leq n < p^{m+1}$. Hence find $2(50!)$, $4(50!)$ and $8(50!)$. 7

(b) Prove that g.c.d of two Fibonacci numbers is also a Fibonacci number . 3

OR

Que.4 (c) Prove that every prime factor of F_n $(n > 2)$ is of the form $2^{n+2}t + 1$, for some integer t . 5

(d) Prove that $S(a) < a\sqrt{a}$, $\forall\; a > 2$. 5

Que.5 (a) If $(a,b) = d$,then prove that general solution of $ax + by = c$ can be written as $x = x_0 + \frac{b}{d}\, t$; $y = y_0 - \frac{a}{d}\, t$, where $t \in \mathbb{Z}$ and $x = x_0$, $y = y_0$ is a particular solution of $ax + by = c$. 3

(b) Find general solution of equation $50x + 45y + 36z = 10$. 3

(c) Find positive integer solution of equation $y - \frac{x+3y}{x+2} = 1$. 4

OR

Que.5 (d) Prove that $x^4 + y^4 = z^2$ has no nonzero positive integer solution . Hence prove that $x^{-4} + y^{-4} = z^{-4}$ has no nonzero positive integer solution . 5

(e) Prove that a general integer solution of $x^2 + y^2 + z^2 = w^2$, $(x, y, z, w) = 1$ is given by $x = (a^2 - b^2 + c^2 - d^2)$, $y = 2ab - 2cd$, $z = 2ad + 2bc$, $w = a^2 + b^2 + c^2 + d^2$. 5

Que.6 (a) State and prove Sun-Tsu theorem . Hence solve the system $x \equiv -2(mod\ 12)$; $x \equiv 6(mod\ 10)$; $x \equiv 1(mod\ 15)$. 7

(b) In usual notation prove that $\sum_{i=0}^{k} \Phi(p^i) = p^k$, where p is prime . 3

OR

Que.6 (c) Prove that $ax + b \equiv 0(mod\ m)$, where$(a, m) = d$, $d > 1$ has solution iff d/b . Also prove that it has d solutions $x_i \equiv a + i\frac{m}{d}(mod\ m)$, $i = 0, 1, 2, \ldots, d-1$, of which $x \equiv a \left(mod\ \frac{m}{d}\right)$ is unique solution of $\frac{a}{d}x + \frac{b}{d} \equiv 0 \left(mod\ \frac{m}{d}\right)$. 5

(d) Prove that Euler's function is multiplicative function . 5

—x—

(3)