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[66/A11] SEAT No.________ **No. of Printed Pages : 2**

SARDAR PATEL UNIVERSITY
B.Sc.(SEMESTER-V)EXAMINATION-2018
October 26, 2018, Friday
10.00 a.m. to 1.00 p.m.
US05CMTH03(MATHEMATICS)(Metric Spaces)

Maximum Marks: 70

Q.1 Choose the correct option in the following questions, mention the correct option in the answerbook. [10]

(1) The set of all cluster points of $\mathbb{N}$ in $\mathbb{R}_d$ is...
(a) $\mathbb{N}$ (b) $\mathbb{Z}$ (c) $\mathbb{R}$ (d) ϕ

(2) The set of all cluster points of $A = \{1, \frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^n}, \ldots\}$ in $\mathbb{R}^1$ is...
(a) $\mathbb{N}$ (b) A (c) $A \cup \{0\}$ (d) $\{0\}$

(3) Let A and B be subsets of a metric space M, then which of the following is true?
(a) $\overline{A} \subset \overline{B} \Rightarrow A \subset B$ (b) $\overline{A} \subset B \Rightarrow A \subset B$
(c) $A \subset \overline{B} \Rightarrow A \subset B$ (d) $A \subset B \Rightarrow \overline{A} = \overline{B}$

(4) Consider $M = [0, 3]$ with discrete metric. Then $B[2; 2] = \ldots$
(a) $(0, 4)$ (b) $\mathbb{R}$ (c) $\{2\}$ (d) $[0, 4]$.

(5) If $E = [0, 5] \cup (4, 7) \subset \mathbb{R}^1$, then $\overline{E}\ldots$?
(a) E (b) $[0, 7]$ (c) $[4, 7]$ (d) $[4, 5]$

(6) Which of the following subset of $\mathbb{R}_d$ is totally bounded?
(a) $[1, 5]$ (b) $(2, 8)$ (c) $\{1, 2, ..., 5^{11}\}$ (d) $\mathbb{N}$

(7) Let $A = [0, 3] \subset \mathbb{R}^1$. Which of the following subset of A is not an open subset of A?
(a) $[0, 2)$ (b) $[0, 2]$ (c) $(1, 2)$ (d) $[0, 3]$

(8) Which of the following subset of $\mathbb{R}^1$ is complete ?
(a) $\mathbb{Q}$ (b) $\{1, 2, ..., 99\}$ (c) $(0, 10]$ (d) $[-5, 1)$

(9) Let A be any subset of $\mathbb{R}_d$, then which of the following is true?
(a) A is connected (b) A is compact
(c) A is bounded (d) A is totally bounded.

(10) For $[0, 2] \subset \mathbb{R}^1$, let $f : [0, 2] \to \mathbb{R}^1$ be a continuous function. Then which of the following is not true?
(a) R_f is connected (b) R_f is compact
(c) f is uniformly continuous (d) none of these

Q.2 Attempt any Ten: [20]

(1) Let $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\rho(x, y) = |x - y|$. Then show that ρ is a metric on $\mathbb{R}$.
(2) Define: (i) Convergence of sequence (ii) Cauchy sequence.
(3) If $\{x_n\}$ is a convergent sequence in $\mathbb{R}_d$, then show that there exist a positive integer N such that $x_N = x_{N+1} = x_{N+2} = \ldots.$
(4) Prove that every subset of $\mathbb{R}_d$ is closed.
(5) Is the union of an infinite number of closed sets is closed? Justify!
(6) Let f be a continuous function from a metric space M_1 onto a metric space M_2. If M_1 is connected, then M_2 is also connected.
(7) Define: (i) Totally bounded set (ii) Complete metric Space.
(8) If (M, ρ) is a complete metric space and A is closed subset of M, Then prove that (A, ρ) is also complete.
(9) Prove that a finite subset A of $\mathbb{R}^1$ is totally bounded.
(10) Define: (i) Compact metric space (ii) Finite intersection property.
(11) Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is uniformly continuous.
(12) Prove that every finite subset of any metric space is compact.

(PTO)

Q.3

(a) Let (M, ρ) be a metric space. If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points of M, then $\{s_n\}_{n=1}^{\infty}$ is Cauchy. Is converse true? Justify! [5]

(b) Let (M, ρ) be a metric space and $a \in M$. Let f and g be real valued functions whose domains are subsets of M. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$, then show that $\lim_{x \to a} [f(x)g(x)] = LN$. [5]

OR

Q.3

(c) Define: Metric space. Let (M, d) be a metric space and let $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$. Then show that d_1 is a metric on M. [5]

(d) Define: Equivalent metrics. If ρ and σ are metrics for M and if there exists $k > 1$ such that $\frac{1}{k}\,\sigma(x, y) \le \rho(x, y) \le k\,\sigma(x, y)$, $\forall\, x, y \in M$ then prove that ρ and σ are equivalent. [5]

Q.4

(a) Let M be a metric space. Then M is connected iff every continuous characteristic function on M is constant. [5]

(b) Let (M, ρ) be a metric space and let A be a proper subset of M. Then the subset G_A of A is an open subset of (A, ρ) iff there exist an open subset G_M of (M, ρ) such that $G_A = A \cap G_M$. [5]

OR

Q.4

(c) Every open subset G of $\mathbb{R}^1$ can be written $G = \bigcup I_n$, where $I_1, I_2, I_3, \ldots$ are a finite number or a countable number of open intervals which are mutually disjoint.(i.e. $I_m \cap I_n = \phi$ if $m \neq n$) [5]

(d) Let (M_1, ρ_1) and (M_2, ρ_2) be metric spaces and let $f : M_1 \to M_2$. Then f is continuous on M_1 if and only if $f^{-1}(F)$ is closed subset of M_1 whenever F is a closed subset of M_2. [5]

Q.5

(a) The subset A of the metric space (M, ρ) is totally bounded iff for every $\epsilon > 0$, A contains a finite subset $\{x_1, x_2, \ldots, x_n\}$ which is ϵ-dense in A. [5]

(b) State and prove Picard's fixed point theorem. [5]

OR

Q.5

(c) State and prove generalized nested interval theorem. [5]

(d) Let (M, ρ) be a metric space. The subset A of M is totally bounded iff every sequence of points of A contains a Cauchy subsequence. [5]

Q.6

(a) Let $(M_1 \rho_1)$ be a compact metric space. If f is continuous function from M_1 into a metric space (M_2, ρ_2), then f is uniformly continuous on M_1. [5]

(b) The metric space M is compact iff whenever $\mathcal{F}$ is a family of closed subsets of M with the finite intersection property, then $\bigcap_{F \in \mathcal{F}} F \neq \phi$. [5]

OR

Q.6

(c) If M is a compact metric space, then prove that M has the Heine-Borel property. [6]

(d) If the real valued function f is continuous on the compact metric space M, then f attains a maximum value at some point of M. Also, f attains a minimum value at some point of M. [4]