[74] SEAT No. ______ No. of Printed Pages : 03

Sardar Patel University, Vallabh Vidyanagar

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Subject : Mathematics US05CMTH03 Max. Marks : 70

Metric Spaces

Date: 15/11/2019, Friday Timing: 10.00 am - 01.00 pm

Instruction : The symbols used in the paper have their usual meaning, unless specified.

Q: 1. Answer the following by choosing correct answers from given choices. **10**

[1] Every Cauchy sequence is
[A] convergent [B] is not always convergent [C] divergent [D] none

[2] A cluster point of the set $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots,\}$ is
[A] 0 [B] 1 [C] 2 [D] 3

[3] For a non-empty set X, a function d defined by $d : X \times X \to R$ where

$$d(x,y) = \begin{cases} 0 \ ; \ \text{if } x \neq y \\ -1 \ ; \ \text{when} x = y \end{cases}$$

is
[A] a usual metric [B] a discrete metric [C] an indiscrete metric [D] none

[4] In a metric space (M, ρ), its subsets M and ϕ are
[A] open but not closed [B] closed but not open
[C] open as well as closed [D] neither open nor closed

[5] In which of the following every closed subset is open also?
[A] R^1 [B] R_d [C] R^2 [D] C

[6] In the metric space $M = [0, 1]$ with usual metric, $B[\frac{1}{4}, \frac{1}{4}] =$
[A] $[0, \frac{1}{4}]$ [B] $(\frac{1}{4}, 1)$ [C] $(0, \frac{1}{2})$ [D] $(0, \frac{1}{4})$

[7] If $|f(x) - f(y)| < 0.9|x - y| \ \ \forall x, y \in R^1$, for a function $f : R^1 \to R^1$ then on R^1, f is
[A] a contraction [B] a homeomorphism [C] discontinuous [D] none

[8] For a subset $A = \{-2, 0, 2\}$ of R^1, $diam(A) =$
[A] 0 [B] -2 [C] 2 [D] 4

[9] If there is atleast one open cover of a metric space M which does not admit a finite sub cover then M
[A] is compact [B] has Heine-Borel property [C] is not compact [D] none

[10] The range of a continuous function f defined on $[1, 2]$ is
[A] unbounded [B] compact [C] disconnected [D] none

Q: 2. Answer TEN of the following. 20

[1] Show that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in R_d then there exists a positive integer N such that $x_N = x_{N+1} = x_{N+2} = ...$

[2] Define $d : \mathbb{R} \longrightarrow \mathbb{R}$ by $d(x, y) = |x^2 - y^2|$. Check whether d is a metric or not.

[3] Show that $\rho : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, defined by $\rho(x, y) = |x - y|$, is a metric on $\mathbb{R}$

[4] Let A and B be subsets of a metric space M. If $A \subset B$, then prove that $\overline{A} \subset \overline{B}$.

[5] Define : (i) Connected Set (ii) Characteristic function

[6] Prove that in any metric space (M, ρ), the set M and ϕ are closed sets.

[7] Define : (i) Bounded set (ii) Diameter of a set

[8] Let $T : [0, \frac{1}{3}] \to [0, \frac{1}{3}]$ be defined by $T(x) = x^2, \ \forall x \in [0, \frac{1}{3}]$. Prove that T is a contraction on $[0, \frac{1}{3}]$.

[9] Prove that every contraction mapping is continuous.

[10] Show that a finite subset of $\mathbb{R}_d$ is compact.

[11] Show that the range of a continuous function, on a compact metric space, is bounded.

[12] Give an example of a function which is one-one, onto, continuous but its inverse is not continuous.

Q: 3 [A] Define Cauchy sequence. Also prove that if $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points in a metric space (M, ρ) then $\{s_n\}_{n=1}^{\infty}$ is Cauchy. Is the converse true? Justify. 5

[B] Let (M, d) be a metric space and let $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$. Then prove that d_1 is a metric on M 5

OR

Q: 3 [A] Let (M, ρ) be a metric space and let a be a point in M. Let f and g be real valued functions whose domains are subsets of M. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = N$ then prove that $\lim_{x \to a} [f(x).g(x)] = \lim_{x \to a} f(x). \lim_{x \to a} g(x)$ 5

[B] prove that a real valued function f is continuous at $a \in R$ iff whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. 5

Q: 4. Let (M, ρ) be a metric space and let A be a subset of M. Then prove that, if A has either one of the following property it has the other.
(a) It is impossible to find nonempty subsets A_1 and A_2 of M such that $A = A_1 \cup A_2$, $\overline{A_1} \cap A_2 = \phi$, $A_1 \cap \overline{A_2} = \phi$.
(b) when (A, ρ) is itself regarded as a metric space, then there is no set except A and ϕ which is both open and closed in (A, ρ). 10

OR

Q: 4 [A] If (M, ρ) is a metric space and A is a proper subset of M then prove that the subset G_A of A is open in (A, ρ) *iff* there exists an open subset G_M of M such that $G_A = A \cap G_M$. 5

[B] Prove that a subset G of the metric space M is open *iff* compliment of G is closed. 5

Q: 5 [A] Let (M, ρ) be a metric space. Then prove that a subset A of M is totally bounded *iff* every sequence of points of A contains a Cauchy subsequence. 5

[B] If the subset A of the metric space (M, ρ) is totally bounded, then prove that A is bounded. 5

OR

Q: 5 [A] If (M, ρ) is a complete metric space and A is a closed subset of M, then prove that (A, ρ) is also complete. 5

[B] State and prove Picard's fixed point theorem. 5

Q: 6 [A] If M is a compact metric space, then prove that M has the Heine-Borel property. 5

[B] If a real valued function f is continuous on a compact metric space M, then prove that f attains the maximum and the minimum values at some points of M. 5

OR

Q: 6 [A] Let f be a one-one, continuous function on a compact metric space (M_1, ρ_1) onto (M_2, ρ_2). Then prove that f^{-1} is continuous and hence f is a homeomorphism of M_1 onto M_2. 5

[B] Prove that if f is a continuous function from a compact metric space M_1 into a metric space M_2 then the range $f(M_1)$ of f is also compact. 5

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