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# [105] Sardar Patel University, Vallabh Vidyanagar

B.Sc. - Semester-V : Examinations : 2020-21 [NC]

Subject : Mathematics US05CMTH03 Max. Marks : 70

Metric Spaces

Date: 28/12/2020, Monday Timing: 02.00 pm - 04.00 pm

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Instruction : The symbols used in the paper have their usual meaning, unless specified.

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Q: 1. Answer the following by choosing correct answers from given choices. [10]

[1] A cluster point of the set $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \}$ is
[A] 0 [B] 1 [C] 2 [D] 3

[2] In $\mathbb{R}_d$, the discrete metric space, $d(0,2) =$
[A] 0 [B] 1 [C] 2 [D] -2

[3] The set of all cluster points of $(1,2)$ is
[A] $[1,2]$ [B] $[1,2)$ [C] $(1,2]$ [D] $(1,2)$

[4] The metric space whose all the subsets are open as well as closed is
[A] $R^1$ [B] $R_d$ [C] $R^2$ [D] none

[5] In the metric space $M = [0,1]$ with usual metric , $B[\frac{1}{4}, 1] =$
[A] $[0,1]$ [B] $[\frac{1}{4}, 1]$ [C] $[0, \frac{1}{4}]$ [D] $(0,1)$

[6] In which of the following every closed subset is open also?
[A] $R^1$ [B] $R_d$ [C] $R^2$ [D] $C$

[7] If a subset $A$ of a metric space $M$ is totally bounded then it is
[A] complete [B] unbounded [C] bounded [D] connected

[8] For a subset $A = \{-2, 0, 2\}$ of $R^1$, $diam(A) =$
[A] 0 [B] -2 [C] 2 [D] 4

[9] The range of a real valued function that is continuous on $[1,2]$ is
[A] not compact [B] not bounded [C] not complete [D] none

[10] If $f$ is a uniformly continuous function then it is
[A] continuous [B] bounded [C] differentiable [D] none

Q: 2. In the following, depending on the type of question, either fill in the blank or answer whether a statement is true false. [08]

[1] The set of cluster points of $Q$ is $Q$. (True or False?)

[2] If $\rho$ is a metric on $M$ then $-2\rho$ is also a metric on $M$. (True or False?)

[3] In $R_d$ we have $B[2,2] = B[3,3]$ . (True or False?)

[4] Every function defind on $R_d$ is discontinuous. (True or False?)

[5] In $R^1$ the subset $[0,1]$ is a totally bounded set. (True or False?)

[6] Every subset of $R_d$ is bounded. (True or False?)

[7] If $f : [0,1] \to R^1$ is continuous on $[0,1]$ then $f([0,1])$ is compact. (True or False?)

[8] If $M$ is compact and $f : M \to R^1$ is continuous on $M$ then $f(M)$ is totally bounded. (True or False?)

Q: 3. Answer ANY TEN of the following. [20]

[1] Define $d : \mathbb{R} \longrightarrow \mathbb{R}$ by $d(x,y) = |x^2 - y^2|$. Check whether $d$ is a metric or not.

[2] Let $d : [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}$ be defined by $d(x,y) = \sin|x-y|$. Show that $d$ is a metric on $[0, \frac{\pi}{2}]$.

[3] Show that $\rho : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ , defined by $\rho(x,y) = |x-y|$, is a metric on $\mathbb{R}$

[4] Prove that in any metric space $(M,\rho)$, the set $M$ and $\phi$ are closed sets.

[5] Give an example of a set $E$ such that $E$ and its compliment are dense in $\mathbb{R}$.

[6] Is the intersection of an infinite number of open sets open? Justify!

[7] Let $T : [0, \frac{1}{3}] \to [0, \frac{1}{3}]$ be defined by $T(x) = x^2, \ \forall x \in [0, \frac{1}{3}]$. Prove that $T$ is a contraction on $[0, \frac{1}{3}]$.

[8] Show that every contraction mapping is continuous

[9] Prove or disprove that the empty set $\phi$ and singletone set are assumed to be connected.

[10] Define : Finite intersection property

[11] Show that a finite subset of $\mathbb{R}_d$ is compact.

[12] Show that the range of a continuous function, on a compact metric space, is bounded.

Q: 4. Attempt ANY FOUR of the following questions. [32]

[1] Let $(M, \rho)$ be a metric space and let $a$ be a point in $M$. Let $f$ and $g$ be real valued functions whose domains are subsets of $M$. If $\lim_{x \longrightarrow a} f(x) = L$ and $\lim_{x \longrightarrow a} g(x) = N$ then prove that
$\lim_{x \longrightarrow a} [f(x) + g(x)] = L + N$

[2] Let $(M, d)$ be a metric space and let $d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$. Then prove that $d_1$ is a metric on $M$

[3] Prove that if $(M, \rho)$ is a metric space then any open sphere in $M$ is an open set.

[4] Prove that a subset $G$ of the metric space $M$ is open *iff* compliment of $G$ is closed.

[5] If the subset $A$ of the metric space $(M, \rho)$ is totally bounded, then prove that $A$ is bounded.

[6] Let $(M, \rho)$ be a metric space. Then prove that a subset $A$ of $M$ is totally bounded *iff* every sequence of points of $A$ contains a Cauchy subsequence.

[7] If $M$ is a compact metric space, then prove that $M$ has the Heine-Borel property.

[8] Let $(M_1 \rho_1)$ be a compact metric space. If $f$ is continuous function from $M_1$ into a metric space $(M_2, \rho_2)$, then prove that $f$ is uniformly continuous on $M_1$.

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