(24)

No of printed pages: 2

Sardar Patel University
Mathematics, B.Sc. Semester V
Tuesday, 12^{th} November 2013
10.30 a.m. to 1.30 p.m.
US05CMTH01 - Real Analysis I

Maximum Marks: 70

Q.1 Choose the correct option for each of the following. [10]

(1) Which of the following is not an ordered field?

(a) $\mathbb{R}$ (b) $\mathbb{Q}$ (c) $\mathbb{C}$ (d) none of these

(2) The infimum of the set $\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$ is

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

(3) Which of the following is not true for $x, y \in \mathbb{R}$?

(a) $|x - y| = |y - x|$ (b) $|x - y| \leq |x| + |y|$ (c) $|x + y| \leq |x| + |y|$ (d) $|x - y| \leq |x| - |y|$

(4) The set $\bigcap_{n=1}^{\infty} [0, \frac{1}{n})$ is

(a) open
(b) closed
(c) both open and closed
(d) neither open nor closed

(5) The derived set of $(0, 1]$ is

(a) $[0, 1]$ (b) $[0, 1)$ (c) $(0, 1]$ (d) $(0, 1)$

(6) Which of the following sets has a limit point?

(a) $\{n + \sin(n) : n \in \mathbb{N}\}$
(b) $\{n + \cos(n) : n \in \mathbb{N}\}$
(c) $\{n + \sin(n) + \cos(n) : n \in \mathbb{N}\}$
(d) $\{\sin(n) + \cos(n) : n \in \mathbb{N}\}$

(7) The value of $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$ is

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Which of the following is true?

(a) f is continuous at every point of $\mathbb{R}$.
(b) f is discontinuous only at 0.
(c) f is discontinuous at every point of $\mathbb{R}$.
(d) f is continuous only at 0.

(9) Which of the following functions is not derivable at 0?

(a) $|x - 1|^2$ (b) $|x|^2$ (c) $|\sin x|$ (d) $|\cos x|$

(10) Which of the following functions is strictly increasing in $(0, \frac{\pi}{2})$?

(a) $\cos x$ (b) $-x^2$ (c) $-x$ (d) $\sin x$

Q.2 Attempt any ***Ten***. [20]

(a) Let A be a non-empty subset of an ordered field F. If the supremum of A exists, then show that it is unique.
(b) State the Archemedian property.
(c) Let $a, x, y \in \mathbb{R}$, and let $a > 0$. Show that $a^{x+y} = a^x a^y$.
(d) Find the interior of the set $\bigcup_{n=1}^{\infty} \left(n - \frac{1}{2^n}, n + \frac{1}{2^n}\right)$
(e) Define limit point of $A \subset \mathbb{R}$.
(f) Find the closure of $\mathbb{Q}$ in $\mathbb{R}$.
(g) Let $f : \mathbb{R} \to \mathbb{R}$, and let $c \in \mathbb{R}$. If $\lim\limits_{x\to c} f(x) = m > 0$, then show that there exist $k > 0$ and $\delta > 0$ such that $f(x) \geq k$ for every $x \in \mathbb{R}$ satisfying $0 < |x - c| < \delta$.
(h) Give an example of a function having discontinuity of first kind at some point.
(i) Evaluate $\lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}$.
(j) Let $f : [0, 1] \to \mathbb{R}$ be $f(x) = x^3$. Show that f is uniformly continuous.
(k) If f is differentiable at c, then show that it is continuous at c.
(l) If $x > y$, then show that $x^3 > y^3$.

Q.3

(a) Define the function $E(x)$. Show that $E(x) = e^x$ for every $x \in \mathbb{R}$. [10]

OR

(b) (i) Show that the real number field is Archemedian. [5]
(ii) Show that there is no rational whose square is 13. [5]

Q.4

(c) (i) Let S be a non-empty bounded subset of $\mathbb{R}$. Let $m = \inf S$. If m is not in S, then show that m is a limit points of S. [5]
(ii) If S is a bounded subset of $\mathbb{R}$, then show that the closure of S is also bounded. [5]

OR

(d) (i) In usual notations show that $C(x + y) = C(x)C(y) - S(x)S(y)$ for $x, y \in \mathbb{R}$. [5]
(ii) Let A be a non-empty subset of $\mathbb{R}$. Show that A is open in $\mathbb{R}$ if and only if it is union of open intervals. [5]

Q.5

(e) Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a)f(b) \leq 0$, then show that there is $c \in [a, b]$ such that $f(c) = 0$. [10]

OR

(f) (i) Let $f : \mathbb{R} \to \mathbb{R}$, and let $a \in \mathbb{R}$. If $\lim_{x\to a} f(x)$ exists, then show that it is unique. [5]
(ii) Let $f : [a, b] \to \mathbb{R}$, and let $x \in [a, b]$. If $\lim_{n\to\infty} f(x_n) = f(x)$ for every sequence $\{x_n\}$ in $[a, b]$ converging to x, then show that f is continuous at x. [5]

Q.6

(g) Let $f : [a, b] \to \mathbb{R}$. Show that f is continuous if and only if it is uniformly continuous. [10]

OR

(h) (i) If $0 < x < 1$, then show that $\log\left(\frac{1+x}{1-x}\right) < 2x\left(1 + \frac{x^2}{3(1-x^2)}\right)$. [5]
(ii) Let $f : [-1, 1] \to \mathbb{R}$ be defined as $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Show that f is derivable at 0. Also find $f'(0)$. [5]

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