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SARDAR PATEL UNIVERSITY

B.Sc. (*SEMESTER – V*) EXAMINATION-2016 (NC)

Thursday, 12th May 2016

10.30 a.m. to 1.30 p.m.

US05CMTH03(Metric Spaces)

Maximum Marks: 70

Q.1 Choose the correct option in the following questions, mention the correct option in the answerbook. [10]

(1) Let d be a metric on M. Then which of the following is not a metric on M?
(a) $d_1(x,\ y) = \min\{1, d(x,\ y)\}$ (b) $d_2(x,\ y) = \max\{1, d(x,\ y)\}$
(c) $d_3(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ (d) $d_4(x,\ y) = 5d(x,\ y)$

(2) The set of all cluster points of $A = \{1, \frac{1}{5}, \frac{1}{5^2}, \ldots, \frac{1}{5^n}, \ldots\}$ is...
(a) $\{0\}$ (b) A (c) $A \cup \{0\}$ (d) $\mathbb{N}$

(3) Let $A = [-2,\ 3] \subset \mathbb{R}^1$. Which of the following subset of A is an open subset of A?
(a) $[0,\ 3]$ (b) $[1,\ 2)$ (c) $(1,\ 3]$ (d) $(0,\ 2]$

(4) Let A and B be subsets of a metric space M, then which of the following is true?
(a) $A \subset \overline{B} \Rightarrow A \subset B$ (b) $A \subset B \Rightarrow \overline{A} = \overline{B}$
(c) $\overline{A} \subset \overline{B} \Rightarrow A \subset B$ (d) $\overline{A} \subset B \Rightarrow A \subset B$

(5) If $E = [1,\ 2] \cup \{3\} \subset \mathbb{R}^1$, then $\overline{E}\ldots$?
(a) $(1,\ 2) \cup \{3\}$ (b) E (c) $[1,\ 2]$ (d) $[1,\ 3]$

(6) Which of the following subset of $\mathbb{R}$ is complete with usual metric of $\mathbb{R}$?
(a) $\mathbb{Q}$ (b) $\{1, 2, ...100\}$ (c) $(2,\ 7]$ (d) $[-1,\ 1)$

(7) Which of the following subset of $\mathbb{R}_d$ is totally bounded?
(a) $[0,\ 5]$ (b) $(-1,\ 1)$ (c) $\{1, 2, ...20^{20}\}$ (d) $\mathbb{N}$

(8) For $[0, 7] \subset \mathbb{R}^1$, let $f : [0, 7] \to \mathbb{R}^1$ be a continuous function. Then which of the following is not true?
(a) R_f is connected (b) R_f is compact (c) R_f is not compact (d) f is bounded

(9) Let $f : [0, \frac{1}{3}] \to [0, \frac{1}{3}]$ be defined by $f(x) = x^2$, then which of the following is not true?
(a) f is a contraction. (b) Range of f is compact
(c) Range of f is connected (d) none of these

(10) Let A be any subset of $\mathbb{R}_d$, then which of the following is true?
(a) A is connected (b) A is compact
(c) A is bounded (d) A is totally bounded

Q.2 Attempt any Ten: [20]

(1) Show that if ρ is a metric for a set M, then so is 4ρ.

(2) If $\{x_n\}$ is a convergent sequence in $\mathbb{R}_d$, then show that there exist a positive integer N such that $x_N = x_{N+1} = x_{N+2} = \cdots$.

(3) Define: (i) Convergence of sequence in metric space (ii) Cauchy sequence.

(4) Let A be an open subset of the metric space M. If $B \subset A$ is open in A, then prove that B is open in M.

(5) Let $M = [-2,\ 3]$ with usual metric. Then find $B[3/4,\ 5/3]$ and $B[7/4,\ 2]$.

(6) Is arbitrary union of closed sets is closed? Justify!

(7) Can a bounded subset of the metric space M be totally bounded? Justify!

(8) If (M, ρ) is a complete metric space and A is closed subset of M, Then prove that (A, ρ) is also complete.

(P.T.O.)

(9) Prove that every contraction mapping is continuous.

(10) Prove that every finite subset of any metric space is compact.

(11) Give an example of a function which is one-one, onto, continuous but its inverse is not continuous.

(12) Let f be a continuous real valued function on $[a, b]$, then prove that f is bounded.

Q.3

(a) Prove that the real valued function f is continuous at $a \in \mathbb{R}^1$ if an only if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers converging to a, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$. [5]

(b) Let (M, ρ) be a metric space. If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points of M, then show that $\{s_n\}_{n=1}^{\infty}$ is Cauchy. Is converse true? Justify! [5]

OR

Q.3

(c) Prove that the real valued function f is continuous at $a \in \mathbb{R}$ if and only if the inverse image under f of any open ball $B[f(a); \varepsilon]$ about $f(a)$ contains an open ball $B[a; \delta]$ about a. [5]

(d) Let (M, d) be a metric space and $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$. Is d_1 a metric on M? Justify! [5]

Q.4

(a) Show that every open subset G of $\mathbb{R}$ can be written $G = \bigcup I_n$, where $I_1, I_2, I_3, \ldots$ are a finite number or a countable number of open intervals which are mutually disjoint. (i.e. $I_m \cap I_n = \phi$ if $m \neq n$) [5]

(b) Let (M_1, ρ_1) and (M_2, ρ_2) be metric spaces and let $f : M_1 \to M_2$. Then prove that f is continuous on M_1 if and only if $f^{-1}(F)$ is closed subset of M_1 whenever F is a closed subset of M_2. [5]

OR

Q.4

(c) Prove that a subset A of $\mathbb{R}$ is connected iff whenever $a \in A$, $b \in B$ with $a < b$, then $c \in A$ for any c such that $a < c < b$. [5]

(d) Let (M, ρ) be a metric space and let A be a proper subset of M. Then prove that a subset G_A of A is an open subset of (A, ρ) iff there exist an open subset G_M of (M, ρ) such that $G_A = A \cap G_M$. [5]

Q.5

(a) Prove that a subset A of the metric space (M, ρ) is totally bounded iff for every $\epsilon > 0$, A contains a finite subset $\{x_1, x_2, \ldots, x_n\}$ which is ϵ-dense in A. [5]

(b) State and prove Picard's Fixed Point theorem. [5]

OR

Q.5

(c) State and prove generalized nested interval theorem. [5]

(d) Prove that a subset A of $\mathbb{R}$ is totally bounded iff A is bounded. [5]

Q.6

(a) Let (M_1, ρ_1) be a compact metric space. If f is continuous function from M_1 into a metric space (M_2, ρ_2), then show that f is uniformly continuous on M_1. [5]

(b) Prove that a metric space M is compact iff whenever $\mathcal{F}$ is a family of closed subsets of M with the finite intersection property, then $\bigcap_{F \in \mathcal{F}} F \neq \phi$. [5]

OR

Q.6

(c) If the metric space M has the Heine-Borel property, then prove that M is compact. [6]

(d) Define Uniform continuity. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous. [4]

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