SEAT No.________ **No. of Printed Pages: 2**

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SARDAR PATEL UNIVERSITY

BSc Sem V Examination, Mathematics

US05CMTH23-Group Theory

Date : 12/11/22 , Saturday   Time : 10-00 am TO 01-00 pm

Q.1 Choose the correct options. (10)

1. The order of 3 in the multiplicative group of non zero rational numbers is_______.

   a.0   b. 1   c. 2   d. infinite

2. In the Klein 4-group $G = \{e, a, b, c\}$, the inverse of an element c is _____

   a. e b. a c. b d. c

3. The set $(M_2(\mathbb{R}), \cdot)$ is not group as _________property of group is not verified.

   a. closure   b. associativity   c. inverse   d. identity

4. For the Euler's $\emptyset$ –function, $\emptyset(5) =$ ______

   a.1   b. 5   c. 4   d. none

5. If $H = -3\mathbb{Z}$ is a subgroup of additive group $G = \mathbb{Z}$ then the index $(G : H) =$ _____

   a.3   b. 5   c. 2   d. 7

6. ______ is generator of the group Z_5^*.

   a. 3   b. 2   c. 3 and 2 both   d. 4

7. If G is a group of order 8 then possible order of its subgroups is _____

   a.3   b. 4   c. 5   d. 7

8. The order of the group S_4 is _____.

   a.4   b. 24   c. 12   d. 20

9. The external direct sum of Z_2 is _____.

   a. Klein 4-group   b. $\mathbb{Z}$ c. $\mathbb{Z}_2$ d. $\mathbb{Q}$

10. A permutation σ is an even permutation if signature of σ is ______.

    a.1   b. -1 c. 2   d. none

Q.2 Answer any TEN. (20)

1) Show that the set $(Z_6, +)$ form a group.
2) In group G, prove that every element of G has unique unit element.
3) State and prove left cancellation law of a Group.
4) Define cyclic group.

(P.T.O.)

5) Define an order of an element.

6) Find order of each element of the group $\{\pm 1, \pm i\}$.

7) Define Kernal of Homomorphism.

8) Let $\theta: G \to G'$ be any homomorphism and e and e' are identities in groups G and G' then prove that $\theta(e) = e'$.

9) Prove that group G is abelian if and only if $i_x = I_G$.

10) Express the cycle $(1 \quad 3 \quad 5 \quad 4 \quad 2)$ as a product of transposition.

11) Prove that every subgroup of abelian group is normal subgroup.

12) Find signature of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$.

Q.3 (a) Let G be a semigroup. Assume that for all $a, b \in G$, the equations $ax = b$ and $ya = b$ have unique solutions in G. Prove that G is a group. **(5)**

(b) Prove that the intersection any number of subgroups of a group G is also a subgroup of group G. **(5)**

OR

Q.3 (a) Let H and K be finite subgroups of group G such that HK is a subgroup of G then prove that $O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$. **(5)**

(b) Prove that every subgroup of cyclic group is also cyclic. **(5)**

Q.4 (a) Prove that any two left cosets of a H in G have the same number of elements. **(5)**

(b) State and prove Lagrange's theorem for a finite group. **(5)**

OR

Q. 4 (a) Let G be a finite cyclic group of order n, then prove that G has $\emptyset(n)$ generators. **(5)**

(b) Let G be a group and $a, b \in G$ such that $ab = ba$. If $O(a) = n, O(b) = m$ with m, n are relatively prime, then prove that $O(ab) = mn$. **(5)**

Q. 5 (a) Define Normal subgroup. Prove that a subgroup H is Normal in group G iff $xH = Hx, \ \forall x \in G$. **(5)**

(b) Prove that a homomorphism $f: G \to G'$ of G to G' is an isomorphism $iff\ Kerf = \{e\}$. **(5)**

OR

Q. 5 (a) State and prove First isomorphism theorem. **(5)**

(b) Let $G = \mathbb{Z}$, $G' = \{z \mid z \in \mathbb{C}, \ |z| = 1\}$, then prove that $G/\mathbb{Z} \cong G'$ **(5)**

Q. 6(a) Define the following terms:

(i)Permutations on n symbols (ii)Transpositions (iii) Cycle and(iv) Signature of Permutation. **(5)**

(b) Prove that S_n, the set of all permutations on n symbols is a group. **(5)**

OR

Q. 6(a) Prove that the mapping $\epsilon: S_n \to \{1, -1\}$ given by $\sigma \to \epsilon\sigma$ is a homomorphism of S_n onto the multiplicative group $\{1, -1\}$ **(5)**

(b) State and prove Cayley's theorem. **(5)**

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